SEVENTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of Harvard University, the seventh annual meeting of the Mathematical Association of America was held at Harvard on Thursday and Friday, December 28 and 29, 1922, in affiliation with The American Association for the Advancement of Science, and in conjunction with the annual meeting of the American Mathematical Society. There were 216 at the sessions, including the following 148 members of the Association:

R. C. ARCHIBALD, Brown University.
C. S. ATCHISON, Washington and Jefferson College.
F. H. BAILEY, Massachusetts Institute of Technology.
L. A. BAUER, Dept. of Terrestrial Magnetism, Washington, D. C.
RALPH BEATLEY, Harvard Graduate School of Education.
W. J. BERRY, Brooklyn Polytechnic Institute.
O. F. H. BERT, Washington and Jefferson College.
E. G. BILL, Dartmouth College.
G. D. BIRKHOFF, Harvard University.
G. A. BLISS, University of Chicago.
S. L. BOOTHROYD, Cornell University.
H. C. BRADLEY, Massachusetts Institute of Technology.
F. E. BRASCH, Chicago, Illinois.
B. H. BROWN, Dartmouth College.
ELEANOR BROWN, Hanover, N. H.
R. E. BRUCE, Boston University.
N. R. BRYAN, University of Maine.
MARGARET BUCHANAN, West Virginia University.
W. G. BULLARD, Syracuse University.
R. W. BURGESS, Brown University.
W. R. BURWELL, Brown University.
W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, University of California.
B. H. CAMP, Wesleyan University.
MILDRED E. CARLEN, Brown University.
F. E. CARR, Oberlin College.
B. E. CARTER, Colby College.
J. W. CLAWSON, Ursinus College.
J. L. COOLIDGE, Harvard University.
LENNIE P. COPELAND, Wellesley College.
J. A. CRAGWALL, Wabash College.
W. L. CRUM, Yale University.
C. H. CURRIER, Brown University.
W. W. DENTON, University of Michigan.
C. E. DIMICK, U. S. Coast Guard Academy.
E. L. DODD, University of Texas.
W. F. DOWNEY, English High School, Boston.
L. H. DUBE, Ottawa University.
L. P. EISENHART, Princeton University.
L. C. EMMONS, Michigan Agricultural College.
T. C. ESTY, Amherst College.
G. C. EVANS, Rice Institute.
G. W. EVANS, Charlestown High School.
FLOYD FIELD, Georgia School of Technology.
J. C. FIELDS, University of Toronto.
C. H. FORSYTH, Dartmouth College.
TOMLINSON FORT, University of Alabama.
PHILIP FRANKLIN, Harvard University.
C. A. GARABEDIAN, Harvard University.
R. W. GARDNER, Eastern Nazarene College.
O. T. GECKELER, Carnegie Institute of Technology.
R. E. GILMAN, Brown University.
O. E. GLENN, University of Pennsylvania.
J. W. GLOVER, University of Michigan.
ALICE B. GOULD, Boston.
W. C. GRAUSTEIN, Harvard University.
C. F. GUMMER, Queen's University.
C. E. HAIGLER, Wentworth Institute, Boston.
W. M. HAMILTON, U. S. Naval Observatory.
E. S. HAMMOND, Bowdoin College.
J. G. HARDY, Williams College.
J. N. HART, University of Maine.
OLIVE C. HAZLETT, Mt. Holyoke College.
H. C. HICKS, Brown University.
H. L. HODGKINS, George Washington University.
E. V. HUNTINGTON, Harvard University.
W. A. HURWITZ, Cornell University.
M. H. INGRAHAM, University of Chicago.
O. D. KELLOGG, Harvard University.
A. E. KENNELLY, Harvard University.
W. D. LAMBERT, U. S. Coast and Geodetic Survey.
GILLIE A. LAREW, Randolph-Macon Woman's College.
H. E. A. LAZOTT, South High School, Worcester.
D. D. LEIB, Connecticut College for Women.
JOSEPH LIPKA, Massachusetts Institute of Technology.
C. C. MACDUFFEE, Princeton University.

MARTHA P. MCGAVOCK, Wellesley College.
H. P. MANNING, Brown University.
T. E. MERGENDAHL, Tufts College.
G. A. MILLER, University of Illinois.
E. B. MODE, Boston University.
E. C. MOLINA, American Telephone and Telegraph Co., New York.
C. L. E. MOORE, Massachusetts Institute of Technology.
C. N. MOORE, University of Cincinnati.
E. H. MOORE, University of Chicago.
F. C. MOORE, Massachusetts Agricultural College.
M. M. S. MORIARTY, Holyoke High School.
R. K. MORLEY, Worcester Polytechnic Institute.
C. C. MORRIS, Ohio State University.
H. M. MORSE, Cornell University.
F. R. MOULTON, University of Chicago.
J. R. MUSSELMAN, Johns Hopkins University.
B. L. NEWKIRK, General Electric Company.
G. D. OLDS, Amherst College.
W. F. OSGOOD, Harvard University.
MARGARET C. PACKER, Hood College.
F. M. PEDERSEN, College of the City of New York.
ANNA J. PELL, Bryn Mawr College.
L. R. PERKINS, Middlebury College.
E. C. PHILLIPS, Woodstock College.
H. B. PHILLIPS, Massachusetts Institute of Technology.
A. D. PITCHER, Western Reserve University.
W. R. RANSOM, Tufts College.
R. G. D. RICHARDSON, Brown University.
L. H. RICE, Massachusetts Institute of Technology.
H. L. RIETZ, University of Iowa.
E. D. ROE, JR., Syracuse University.
W. H. ROEVER, Washington University.
GEORGE RUTLEDGE, Massachusetts Institute of Technology.
F. H. SAFFORD, University of Pennsylvania.
RALEIGH SCHORLING, Lincoln School, New York.
H. C. SHAUB, Harvard University.
W. G. SIMON, Western Reserve University.
MARY E. SINCLAIR, Oberlin College.
H. E. SLAUGHT, University of Chicago.
H. L. SLOBIN, New Hampshire College.
CLARA E. SMITH, Wellesley College.
D. E. SMITH, Columbia University.
SARAH E. SMITH, Mt. Holyoke College.
MAY J. SPERRY, Syracuse University.
J. M. STETSON, University of Illinois.
K. D. SWARTZEL, University of Pittsburgh.
J. S. TAYLOR, Massachusetts Institute of Technology.
TELESFORO TIENZO, University of the Philippines.
MARIAN M. TORREY, University of West Virginia.
J. I. TRACEY, Yale University.
A. B. TURNER, College of the City of New York.
H. S. UHLER, Yale University.
E. B. VAN VLECK, University of Wisconsin.
OSWALD VEBLEN, Princeton University.
ROXANA H. VIVIAN, Wellesley College.
J. L. WALSH, Harvard University.
A. G. WEBSTER, Clark University.
J. H. M. WEDDERBURN, Princeton University.
MARY E. WELLS, Vassar College.
V. H. WELLS, Williams College.
A. H. WHEELER, North High School, Worcester.
D. E. WHITFORD, University of Rochester.
J. K. WHITTEMORE, Yale University.
C. E. WILDER, Dartmouth College.
F. B. WILLIAMS, Clark University.
RUBY WILLIS, Wellesley College.
E. B. WILSON, Harvard School of Public Health.
RUTH G. WOOD, Smith College.
F. S. WOODS, Massachusetts Institute of Technology.
JESSICA M. YOUNG, Washington University.
J. W. YOUNG, Dartmouth College.
S. D. ZELDIN, Massachusetts Institute of Technology.

The various meetings of the American Association for the Advancement of Science were held throughout the week, mainly at the buildings of the Massachusetts Institute of Technology, beginning Tuesday evening with the address of the retiring President, Professor E. H. Moore, on the subject "What is a Number System?" The manifold programs in the meetings of the various Sections of the American Association and of the affiliated Societies furnished a wealth of opportunity for exercising one's choice. The majority of mathematicians stayed in comfortable rooms in the Harvard dormitories and were thus within easy reach of the places of meeting, and of the Harvard Union and other convenient dining halls. While the formal programs of both mathematical

organizations were held at Harvard University, a close coöperation between the staff of Harvard and of the Massachusetts Institute of Technology gave the attending members a chance to enjoy the hospitality of both institutions. It was thus possible between the two morning sessions of the Society to enjoy the Symposium on Space and Time, under the auspices of the Society and Section A, Wednesday afternoon at the Technology buildings, at which time Professors G. D. BIRKHOFF, P. W. BRIDGMAN, and HARLOW SHAPLEY, all of Harvard University, spoke respectively, on "The logic of Space and Time," "The physical meaning of Space and Time," and "The astronomical measures of Space and Time." On Wednesday evening various groups attended the dinner for visiting women, the trip to the Harvard Astronomical Observatory, a private dinner for Harvard men, the Sigma Xi dinner, and a smoker in the Harvard Union. On Thursday which was "Harvard Day," for the American Association and the affiliated societies, a complimentary luncheon was served by Harvard University at Memorial Hall. A pleasant occasion at the close of each afternoon session was the serving of tea at Phillips Brooks house by the ladies of the Harvard Faculty, under the chairmanship of Mrs. Graustein; this, with the other provisions made by the ladies for the entertainment of the visitors, was recognized in a resolution adopted at the closing session, on motion by Professor Clara Smith. At this same session Professor Floyd Field offered a motion, which was heartily adopted, recognizing the unusual interest aroused by the programs, and expressing the thanks of the Association to the departments at Harvard and "Tech" for the hearty reception and careful arrangements; to Professors Coolidge, Kellogg and Lipka of the Committee on Arrangements for their efficiency, courtesy, and unfailing good humor in their busy task; to Professors Huntington, Copeland and Currier of the Program Committee for the presentation of the program of such a wide range of interest to all, and to the officers of the Association for their administration of the affairs of the Association for the past year.

The joint dinner of the two organizations and Section A was held Thursday evening at the Walker Memorial Building of the Institute of Technology, with 135 persons present, Professor David Eugene Smith being the genial toastmaster. Introducing four men, as he said, to give an account of the three periods of mathematical history in America, he called first upon Professor Cajori, who spoke on the early period of mathematics in America, describing what was probably the first controversy on a scientific subject, and describing the earliest permanent observatory, established at Bogota in 1803. Professor Van Vleck spoke on the "middle ages" in American mathematics, telling how this central period began to make strong provision for the existence of suitable journals, of acceptable centers of influence and, most of all, of an atmosphere favorable for mathematical study; how these ends were provided through the *American Journal*, as the first able mathematical publication, Johns Hopkins University as the first strong center, followed by the establishment of those at Clark University and at Chicago, the establishment of the New York Mathematical Society in 1891 and its extension in 1894 into the American Mathematical Society; he appealed

especially to the younger men in American mathematics to see to it that it shall be possible for us to say:—"The best is yet to be." Professor Eisenhart spoke of the large and influential work done by the American Mathematical Society and its journals, and of the difficult financial problem that is now being faced in conducting the Transactions. Professor Slaught, called upon as a representative of the Association, spoke of the founding of the Mathematical Association of America, as lying within the more recent period of American mathematics, of the widened opportunity on the one hand for edifying receptivity in matters of collegiate mathematics, and on the other hand for fruitful activity represented, for example, in the pages of the MONTHLY, in the summer and annual meetings, and in at least ten section meetings each year, stressing above all the opportunity to help larger and larger numbers of our teachers and students to begin to mount the ladder of research through the furnishing of the lower rounds of this ladder. Professor Smith expressed very happily the sentiments of those present toward "Old Harvard" and "Tech," and this was reinforced by the visitors by a rising vote.

Professor G. A. Miller presided at the joint session on Thursday afternoon. President Archibald presided for most of the Friday morning session, calling Professor Eisenhart, one of the newly elected vice-presidents, to the chair for the rest of the day.

The following program was given, abstracts of part of these papers being given with numbers corresponding to the numbers in the list of titles.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION.

(1) "Reduction of singularities of plane curves by birational transformation" by Professor G. A. BLISS, University of Chicago, retiring president of the American Mathematical Society.

(2) "The grafting of the theory of limits on the calculus of Leibniz" by Professor FLORIAN CAJORI, University of California, representing the Mathematical Association of America.

(3) "Geometry and physics" by Professor OSWALD VEBLEN, Princeton University, retiring vice-president of Section A.

1. President Bliss's address is to be printed in the *Bulletin of the American Mathematical Society*.

2. Professor Cajori's address will appear shortly in this MONTHLY.

3. Professor Veblen's address appeared in *Science* for February 2, 1923.

SESSION OF THE ASSOCIATION.

(4) "Period of the bifilar pendulum for finite amplitudes" by Professor H. S. UHLER, Yale University.

(5) "Skew squares" by Professor W. H. ECHOLS, University of Virginia.

(6) "On the averaging of grades" by Professor C. F. GUMMER, Queen's University.

(7) "Mathematics at Oxford and the Ph.D. degree" by Professor W. R. BURWELL, Brown University.

(8) "Some unsolved problems in the theory of sampling" by Professor B. H. CAMP, Wesleyan University.

(9) "Some unsolved problems in solid geometry" by Professor J. L. COOLIDGE, Harvard University.

4. The primary object of Professor Uhler's investigation was to derive a convergent series for the period of the bifilar pendulum corresponding to any feasible finite amplitude of vibration. The material is conveniently divided in three parts. In part (*a*) attention is called to the unsatisfactory state of the literature of the problem. Then a set of rigorous equations are given which lead to a hyperelliptic integral as the formula for the period. In part (*b*) this integral is expanded into two related series which depend upon a fundamental series the law of development of which is evident. It is shown that the second and third approximation terms of the two final series may be either independently or simultaneously positive, or zero, or negative. Numerical illustrations are then presented. In part (*c*) the domain of convergence of the series is discussed and shown to depend upon points lying between certain straight and parabolic lines in a Cartesian diagram.

This paper appeared in the *Journal of the Optical Society of America* for March, 1923.

Professor Slaught stated that Professor W. D. MacMillan has treated a similar problem where the rod is suspended in the manner of a bifilar pendulum, but where the motion is not limited to mere twisting, the rod being put in motion from any position in which it may be drawn off from its vertical position of rest. The motion of the rod when released is of an amazing nature.

Professor Huntington entered a plea for the construction of a table of elliptic integrals for all values of the parameter, saying that such a table would be of great usefulness in this and similar problems.

5. In the absence of Professor Echols, due to illness in the family, Professor Graustein gave a very acceptable presentation of the paper. This will be printed in an early issue of the MONTHLY.

In the discussion Professors Young and Clara Smith criticized the use of the term "skew square," which usually suggests a quadrilateral not lying in one plane; and Professor Archibald stated that several French writers had used the term pseudo-square for this kind of figure.

6. Examining boards often wish to arrange in some kind of order of merit candidates who have been examined in a number of different subjects; and in several other connections the idea of "average grade" is more or less directly present. Of the various functions $f(x_1, x_2, \cdots, x_n)$ that may be arbitrarily selected to represent such an average for grades $x_1, x_2, \cdots, x_n$, of equal weight, it may reasonably be demanded that (*a*) $f(x_1, x_2, \cdots)$ be symmetric, (*b*) $f(x_1, x_2, \cdots, x_n) = x$, (*c*) $f(kx_1, kx_2, \cdots) = kf(x_1, x_2, \cdots)$ for positive k, (*d*) continuous first partial derivatives exist and remain positive for all positive x's.

The simplest solution meeting these requirements would seem to be that in which f is the quotient of a homogeneous quadratic by a homogeneous linear form, the formula being $f = a + cd/a$, where a is the arithmetic mean, d the mean square of the deviation from a, and c a constant open to choice, which will be taken negative if it is desired to put a premium on equality of performance. To satisfy the latter part of (*d*) c must lie between $-1/(n-1)$ and $(n-1)/(2n-1)$.

A formula in some ways more convenient is available if (*d*) is so far relaxed as to allow the derivatives to become discontinuous and non-existent at certain places, the discontinuity being preferably not too abrupt; namely, $f = m_1y_1 + m_2y_2 + \cdots + m_ny_n$, where y_i is the ith greatest x and the m's are constants. If we agree that a additional marks equally divided among the r best subjects shall be counterbalanced by the loss of b marks equally divided between the $n - r$ lowest, whatever r may be, we are able to determine the m's in terms of the arbitrary ratio a/b. The result may be written $f = y_n + s_{n-1}(y_{n-1} - y_n) + \cdots + s_1(y_1 - y_2)$, where $s_i = m_1 + m_2 + \cdots + m_i = ib/(na - ia + ib)$. This formula is very easy to use. To favor equality a must be greater than b. Various modifications of the method suggest themselves.

In the discussion of Professor Gummer's paper, Professor Dodd referred to Schimmack's axioms in the *Mathematische Annalen*, volume 68, 1910, pp. 125–132, pointing out that the assumption of the existence of the derivatives rules out the use of the median in making an average. He referred also to the treatment of a somewhat similar question in a paper by Daniell in the *American Journal of Mathematics*, volume 42, 1920, 222–236.

7. Professor Burwell's paper will appear in an early issue of the MONTHLY.

8. Professor Camp confined his attention to simple sampling from distributions involving but one variate. The general question to be answered is: How good is a sample? The answers may be discussed under three heads, as follows. (*a*) Sampling for a mean value. What is the probability that the mean of the sample will lie within a prescribed amount of the mean of the parent distribution? The solution of this problem has been published only for the special case where the parent distribution is normal, or sufficiently near to normal to warrant the assumption that the "curve of means" is normal. It is, however, probable that the general case may be solved by a combination of some of the theory of Charlier with certain relations developed by the Pearsonian school. (*b*) Sampling for a standard deviation or other statistic. Here again only the special case where the parent distribution is normal has been adequately treated. (*c*) Sampling for a frequency curve. What is the probability that the curve found by the sample will lie within a prescribed belt of the ideal? The classic answer to this question is given by K. Pearson's χ test. Edgeworth's criticism of the random use of this test is important and may be found in his article on "Probability" in the *Encyclopædia Britannica*.

The relatively simple questions considered above suggest more difficult and more practical ones, for which, save in highly specialized instances, there exist no answers. If one starts with a sample, instead of with a parent distribution,

one has the problem of inverse probability, called by K. Pearson "the fundamental problem of practical statistics," with the various special cases (*a*), (*b*), (*c*), as above. Finally there is the broad field of generalization to functions of more than one variable, involving the properties of correlation surfaces rather than of frequency curves.

Professor Rietz called attention to the fact that in the *Journal of the Royal Statistical Society* for January, 1922, Yule and Fisher presented a modification of Pearson's criterion of goodness of fit of theory and observation by showing that the usual n with which to enter the tables of Elderton should be one more than the number of degrees of freedom of the frequency system.

9. The paper by Professor Coolidge adds to the series of valuable papers which have already appeared on the programs of the Association as giving suggestions as to possible problems in research in familiar fields. This paper will appear in full in an early issue of the MONTHLY.

SYMPOSIUM ON MATHEMATICAL STATISTICS.

(10) "The subject matter of a course in mathematical statistics" by Professor H. L. RIETZ, University of Iowa.

(11) "Time series of economic statistics: Their fluctuation and correlation" by W. M. PERSONS, professor of economics, Harvard University, and editor of the *Review of Economic Statistics*.

(12) "Some fundamental concepts of the calculus of mass variation and their relation to practical problems" by ARNE FISHER, author of *The Mathematical Theory of Probabilities and Its Application to Frequency Curves and Statistical Methods*.

10. Professor Rietz's outline of a course in mathematical statistics will be printed in a later issue of the MONTHLY.

11. Time series differ in important respects from other series of economic statistics. The items of a time series (1) must be defined for selected time units, (2) are ordered in time, and (3) during adjacent time-intervals are affected by the same or similar influences. These peculiarities give rise to characteristic types of fluctuations and to special correlation problems.

Four types of variations are to be found in most time series. These are: the long-time movement or secular trend; the seasonal variation; the cyclical movement connected with the ebb and flow of business; and irregular fluctuations resulting from panics, strikes, etc. In order to isolate the cyclical movement, so far as may be, it is necessary to measure the secular trend and seasonal variation. In Professor Persons' paper attention was directed to the problems of measuring seasonal variation and of finding the significant relations between series corrected for secular trend and seasonal variation. Monthly rates on commercial paper, 1866–1922, are used to illustrate the methods of handling the problems.

The obvious method of finding seasonal variation is to eliminate secular trend and then take arithmetic means of corresponding months (Januaries,

Februaries, etc.). This method works well for long series or parts of series *which are not subject to large irregular fluctuations*, but even in these cases it has the defect that the *degree of regularity* of the supposed seasonal movement is not set forth. The method works badly for the great majority of economic series, because such series contain many non-seasonal fluctuations, highly irregular both as to size and time of occurrence, which greatly distort the arithmetic means of the items. In other words, the actual data with which the economist has to deal do not follow the ideal functional relations, which would lead to the use of the arithmetic means of all the items for corresponding months for determining the seasonal variation. (The assumptions made by W. L. Hart, for instance, in his paper on "The method of monthly means for determination of a seasonal variation," *Journal of the American Statistical Association*, September, 1922, cannot be retained in the great majority of cases.)

A method of measuring seasonal variation which has been found to work well in practice (see the *Review of Economic Statistics*, January 1919, pp. 18–31) is the following: find the link relatives of the original items; tabulate the link relatives for January, February, etc., as frequency distributions; find the medians (or, if preferred, the averages of the 3, 4 or more central items) of the link relatives; chain the medians (or selected average of central items) to secure a fixed base; distribute the discrepancy between the beginning and end of the chain relatives among all the members of the chain (the distribution may be either geometric or arithmetic).

The chief advantages of the method just outlined are: (*a*) the use of frequency tables of link relatives enables one to ascertain the degree of regularity of seasonal changes; (*b*) the use of the median (or average of central items) minimizes the influence of non-seasonal, irregular fluctuations; (*c*) the entire procedure makes it possible to utilize series of items covering long intervals, not entirely homogeneous, and subject to varying cyclical and irregular fluctuations.

Comparison of the indices of seasonal variation computed by the link-relative-median method, and the arithmetic average method for rates on commercial paper for the intervals 1866–73, 1874–89, 1890–99, and 1900–13 shows: significant divergence for intervals with marked irregular disturbances, such as 1866–73 and 1890–99; and practically identical results *when the years of great disturbance, such as 1873, 1890, 1893, 1896, 1898, 1907*, and *1914 are excluded from the arithmetic averages.*

The frequent occurrence of large irregular fluctuations in economic data greatly affects the Pearsonian coefficients of correlation for pairs of series, even when the items are corrected for seasonal variation and secular trend. For instance, for Bradstreet's prices and pig-iron production, bimonthly, 1903–1914, the coefficient of correlation for concurrent items (65 in number) is + .73, and omitting 4 items in 1908 is only + .66; for Bradstreet's prices and New York clearings six months later the coefficient is + .60, and omitting 4 items is only + .50. Coefficients of correlation possess different significance for economic data than for individual measurements and great caution must be used in the

interpretation of such coefficients. Comparison of the graphs of economic series, duly corrected for secular trend and seasonal variation and expressed in units of standard deviation, gives much more information concerning existing relationships than is given by coefficients of correlation. The irregular fluctuations of time series of economic statistics not only must be considered and allowed for in computing indices of seasonal variation and in interpreting coefficients of correlation but also in applying periodogram analysis to such series.

A question by Professor Burgess brought the answer that the foregoing is a method of "red-handed empiricism," used with the purpose of trying to separate the terms of the first three types and then to analyze the irregular fluctuations.

12. Mr. Fisher in his address emphasized the fact that the dominant feature of our material world is that of variation. The old religious philosophies of our Aryan ancestors emphasized the presence of inequality and variation in nature; without variation evolution would be impossible according to the gospel as preached by the great Buddha. Modern science and especially the far-reaching studies inaugurated by Charles Darwin have again brought the question of variation to the forefront. Darwin was the first to point out the definite presence of variation among a large group of similar objects (a species). While Darwin was content to establish the paramount feature of variation in animal and plant life and to outline its general philosophical contours, he did not attempt to subject it to purely quantitative measurement.

This latter aspect of the problem of variation is primarily of a statistical, *i.e.*, of a purely mathematical nature. Statistics, according to the speaker, might properly be defined as the calculus of mass variations. The question then arose how this particular calculus should be formulated and taught. Should it be presented as a purely empirical discipline, or should it be made subject to a more rigorous mathematical method of attack and like all mathematical epistemology be made to rest on a purely *a priori* foundation?

The empirical method of investigation is, however, very limited in scope and apparently rests upon the fallacious doctrine of John Stuart Mill that it was possible to derive the corpus of human knowledge by what Mill termed an "*inductio per simplicem enumerationem.*" This method has especially been used by the German school of statistics under von Mayr and his disciples. Until comparatively recent years this school has been the only one followed in statistical work here in America. Its chief exponents are found among such writers as Mayo-Smith, Willcox, Hoffman, Dublinsky and Secrist. Unfortunately, in the opinion of the speaker, most of such writings often degenerate into mere verbiage. One recent American writer in his text on statistics spends more than fifty pages in the discussion of such simple statistical characteristics as the mean and the dispersion. By the employment of elementary mathematical theorems from the calculus of probabilities the same amount of information could be compressed into a few simple formulas, not occupying more than a single book page. All that such authors spent pages upon pages in telling to their readers could be seen in a mere glance at the empirical tables themselves.

The reason for this is chiefly to be found in the fact that the method of investigation as employed by the authors referred to above is really no method at all. The fallacy of these authors is to be found in the fact that they mistake a mere collection of observations (statistical data) for a statistical analysis.

The keen mind of the great Greek scholars (Aryan scholars) always laid stress upon their doctrine that it was the relation *between* the facts which was the important thing to unearth. Modern astronomy offers a striking illustration of this statement. For centuries the Egyptians, the Babylonians, the Greeks, the Hindus, and the Arabs had made mass observations on the celestial objects. An immense mass of statistical data had been gathered and later on augmented and arranged in tabular form by the Danish astronomer Tycho Brahe. Yet the mechanical laws governing our universe were not discovered before Kepler. Newton and Laplace had subjected these statistical data to a mathematical investigation. It was then found that apparently complicated phenomena in reality were subject to simple mathematical laws. Our present day so-called sciences of economics and sociology are still waiting for their Newtons and Laplaces before they really are worthy of the name of science.

What made astronomical science possible was the mathematical tools as furnished by the differential and the integral calculus, a mathematical discipline which rests upon a purely *a priori* foundation, and which made it possible to subject the collected astronomical statistics to a quantitative analysis. Have we now a similar calculus in the case of mass variation? While the speaker was of the opinion that we do not possess a complete set of tools and many new implements have still to be forged, he would, on the other hand, emphasize that we have at least several good tools in the theory of probabilities. Statistical analysis and mathematical statistics must necessarily, in his opinion, be an application and an extension of the theory of probability.

In this particular mathematical discipline the great work of Laplace remains still the greatest contribution. Laplace's theory has become especially useful through the additional researches by the Scandinavian statisticians and mathematicians Gram, Thiele, Jörgensen, Charlier, Wicksell and Westergaard. The speaker contrasted these methods with those introduced by the English mathematician and biometrician Karl Pearson.

Pearson's theory of frequency distribution is derived from the differential equation

$$\frac{d \log y}{dx} = \frac{x + a}{b_0 + b_1 x + b_2 x^2}.$$

This equation gives rise to a great variety of frequency curves. It is, however, more or less a graduation formula based upon rather empirical foundations, one disadvantage of Pearson's formula being that it does not allow of more than four constants, which often is insufficient to represent frequency distributions.

The Scandinavian authors who follow Laplace have based their theory of frequency curves on the following integral equation:

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{\lambda_1\omega}{1!}+\frac{\lambda_2\omega^2}{2!}+\frac{\lambda_3\omega^3}{3!}+\cdots} e^{x\omega i}d\omega,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the semi-invariants of Thiele, defined by the relation

$$e^{\frac{\lambda_1\omega}{1!}+\frac{\lambda_2\omega^2}{2!}+\frac{\lambda_3\omega^3}{3!}+\cdots}\int_{-\infty}^{+\infty}\varphi(x)dx = \int_{-\infty}^{+\infty} e^{x\omega}\varphi(x)dx.$$

If all semi-invariants above λ_2 vanish, we have

$$\varphi(x) = \frac{1}{\sqrt{2\pi\lambda_2}} e^{-\frac{(x-\lambda_1)^2}{2\lambda_2}}$$

or the usual normal probability curve of Laplace.

As shown by Gram and Charlier any frequency function which, together with its derivatives, vanishes for $x = -\infty$ and $x = +\infty$ can be represented by the following expansion:

$$F(x) = \varphi(x) + c_3\varphi'''(x) + c_4\varphi''''(x) + \cdots,$$

where

$$c_3 = -\frac{\lambda_3}{3!}, \qquad c_4 = \frac{\lambda_4}{4!}, \qquad \cdots.$$

The speaker also mentioned the expansion of frequency series by means of the Poisson-Charlier series and the logarithmic transformation as discussed in his own works on probability. As a very general expansion the following function introduced by Bachelier:

$$F(x, t) = \frac{1}{\sqrt{2\pi}\ \varphi(t)} e^{-\frac{[x-\varphi(t)]^2}{2\varphi(t)^2}}$$

was mentioned as being very powerful.

Mr. Fisher claimed that this function was especially useful in economic statistics and afforded a far more useful tool of investigation than the somewhat looser mathematical methods employed by the Harvard Bureau of Economics. The Bachelier and Gram methods might, for instance, be used to solve the following problem: What is the probability that a certain stock or bond will be quoted at a price x at time t on the stock exchange? The speaker showed an actual application he himself had made in the matter of forecasting three months in advance the weekly quotations of a certain gilt-edge stock on the Copenhagen Stock Exchange. During the year of 1922 the lowest value of this stock had been 196 and the highest value 243. The greatest difference between any weekly forecast and the prices actually quoted had been 4 per cent. for one of the first weeks of March. The total accumulated percentage differences from January to December 15, 1922, were less than one tenth of one per cent.

The speaker criticized the investigations by various economists of the so-

called business cycles as being the work of mathematical dilettantes. He mentioned an eminent economist as having said at a recent statistical meeting "that the economists would like to have the mathematicians express their formulas in such a manner that the man in the street could understand what they (the mathematicians) were talking about." The danger of such a spurious argument was that it might equally well be applied to the economists themselves. The immortal answer of Euclid to the Ptolemæan emperor: "There is no royal road in mathematics" was in the opinion of the speaker as true today as it was in the days of antiquity and applied equally well to our present-day analysis in economics and sociology.

Professor J. R. Musselman, of Johns Hopkins University, described the elementary and advanced course in mathematical statistics as given at Johns Hopkins, the former being offered to those who have had the elements of calculus and analytic geometry, and including the topics of permutations and combinations, elementary probability, the use and limitations of the theory of the correlation coefficient, the theory of sampling and curve fitting; time is also available to include the theory of least squares.

Professor Franz Boas, of the department of anthropology, Columbia University, spoke of the great need of instruction in this field for students in various departments of biology. We try too often to treat the theory of averages, standard deviation, correlation and similar topics in a purely formal way; we need rather to give the student of biometry and analogous fields a clear understanding of the subject, and the ability to come to a clear grasp of just what the problems under consideration actually are. This aspect of the subject is hardly recognized in the existing literature.

Professor Theodore H. Brown, of the School of Business, Columbia University, stated that he continually met with New York business men who claim on various grounds to be statisticians. Too frequently men are put to work on statistical matters without an adequate, or at least a ready, hold of the necessary mathematics. There is a large class of students who are hungry for training in this line, and who might be reached through properly prepared texts.

Dr. Herbert A. Toops, of the Institute of Educational Research, Teachers College, Columbia University, pointed out the need that educators have for statistical work, and stated that they ordinarily depend in a groping way on difficult texts in statistics. The Institute aims to develop new methods of research. Presuming a basis of college algebra and a high degree of intelligence, the Institute does, through shortened methods, enable their students to carry on tabulations and similar operations, and also carries out extended work with a few who by further study shall be able to go back to the original sources. It is quite important in this field of statistics to develop really practical statisticians, to give, for example, practice in computing a percentile from both ends in order to check the result, in the use of computing machines, etc.

Professor Huntington called attention to the fact that the National Research Council has appointed a Committee on the mathematical analysis of statistics,

of which Professor Rietz is the chairman, and said that a handbook will be published by this committee within a year, through one of the well-known publishing houses, a book to which those present at this program, and others interested in the subject, will look forward with eagerness.

MEETING OF THE BOARD OF TRUSTEES OF THE ASSOCIATION.

Ten members of the Board were present at the meeting.

The following thirty-one persons and one institution, on applications duly certified, were elected to membership:

To individual membership.

OLIVE ATWOOD, A.B. (Carleton Coll.). Instr., Agricultural Coll., Agricultural College, N. Dak.
ELEANOR BROWN, (Mrs. B. H.), Ph.D. (Radcliffe). Hanover, N. H.
R. D. BURDICK, A.M. (Columbia). Instr., Coll. of City of New York, New York, N. Y.
H. W. CHANDLER, M.S. (Iowa). Instr., Univ. of Minnesota, Minneapolis, Minn.
E. E. COLYER, A.M. (Kansas). Prof., Fort Hays Kansas State Normal School, Hays, Kansas.
W. H. CRAMBLET, Ph.D. (Yale). Prof., Bethany Coll., Bethany, W. Va.
L. A. V. DECLEENE, (Rev.), A.M. (Catholic Univ.). Graduate student, Catholic Univ., Washington, D. C.
LALIT MOHAN DEY. Govt. official, Calcutta, Shambazar P.O., India.
LAURA DUERNER, B.S. (Oregon). Instr., N. Dak. Agric. Coll., Fargo, N. Dak.
J. T. ERWIN, A.M. (Vanderbilt). Prof., George Washington Univ., Washington, D. C.
H. W. FICKEN, B.S. (Coll. City of N. Y.). Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
S. E. FIELD, A.M. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.
R. W. GARDNER, B.S. (Olivet). Head of dept. of math., Eastern Nazarene College, Wollaston, Mass.
O. T. GECKELER, A.B. (Indiana). Asso. prof. and acting head of dept., Carnegie Inst. of Tech., Pittsburgh, Pa.
MARTHA HILDEBRANDT, Ph.B. (Chicago). Teacher, Proviso High School, Maywood, Ill.
F. W. JOHN, M.E. (Cornell). Instr., Washington Square Coll., N. Y. Univ., New York, N. Y.
H. E. A. LAZOTT, A.B. (Boston Univ.). Instr., South High School, Worcester, Mass.
E. A. LELACHEUR, A.B. (Valparaiso). Computer, U. S. Coast and Geodetic Survey, Washington, D. C.
W. V. LOVITT, Ph.D. (Chicago). Prof., Colorado Coll., Colorado Springs, Colo.
W. E. MACDONALD, A.M. (Harvard). Prof., Canton Christian Coll., Canton, China.
WILLIAM MARSHALL, Ph.D. (Zurich). Prof., Purdue Univ., Lafayette, Ind.
A. E. MEDER, Jr., A.B. (Columbia). Asst., Columbia Univ., New York, N. Y.
HELEN MOON, M.S. (Iowa). Graduate student, Univ. of Iowa, Iowa City, Ia.
L. T. MOORE, B.S. (Emory Univ.). Asst., Johns Hopkins Univ., Baltimore, Md.
R. H. MORTIMORE, A.B. (Iowa). Instr., Graceland Coll., Lamoni, Ia.
P. H. NYGAARD, A.B. (St. Olaf). Teaching Asst., Univ. of Minn., Minneapolis, Minn.
MARGARET C. PACKER, A.M. (Brown). Instr., math. and physics, Hood College, Frederick, Md.
N. A. PATTILLO, Ph.D. (Johns Hopkins). Dean and prof. of math., Randolph-Macon Woman's Coll., Lynchburg, Va.
J. E. REDDEN, A.B. (Furman Univ.), B.S. in C.E. (Clemson Coll.). Asst. prof., John Tarleton Agric. Coll., Stephenville, Tex.
ARTHUR TILLY. Instr., Washington Square Coll., N. Y. Univ., New York, N. Y.
MARY RUTH WHALEY, A.B. (Smith). Instr., St. Agnes School, Albany, N. Y.

To Institutional Membership.

WESTERN COLLEGE FOR WOMEN, Oxford, Ohio, Prof. Harriet E. Schoonmaker, Official representative.

The following have been appointed associate editors of the MONTHLY for the year 1923:

N. H. ANNING	B. F. FINKEL	R. B. MCCLENON
R. W. BURGESS	TOMLINSON FORT	C. N. MILLS
E. L. DODD	D. C. GILLESPIE	F. B. MURNAGHAN
OTTO DUNKEL	C. F. GUMMER	D. E. SMITH

Professor H. L. RIETZ was appointed to be the representative of the Association in the Division of Physical Sciences, National Research Council, for the three-year term beginning June 11, 1923, in succession to Professor E. R. Hedrick.

It was voted to hold the annual meeting next December at the University of Cincinnati in affiliation with the American Association. The incoming President was empowered to appoint the necessary committees for the summer meeting, and an announcement of the place and time will be made as early as possible. Professor A. D. PITCHER and the Secretary were appointed as representatives of the Association on the Council of the American Association for the year 1923.

Professor R. C. ARCHIBALD was appointed to the vacancy for the term ending January 1925, caused by the election of Professor Eisenhart as a vice-president.

The Trustees appointed the following persons for the year 1923:

Secretary-Treasurer: W. D. CAIRNS.

Librarian: L. C. KARPINSKI.

Members on Committee on Official Journal: J. L. COOLIDGE, W. B. FORD, and H. E. SLAUGHT.

Assistant Secretary: C. H. YEATON.

Assistant Librarian: MARY E. SINCLAIR.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership by the Board. He reported also the death of the following members:

FREDERICK ANDEREGG, Professor of mathematics, Emeritus, Oberlin College (October 9, 1922);

W. W. BEMAN, Professor of mathematics, University of Michigan (January 18, 1922);

C. L. BOUTON, Associate professor of mathematics, Harvard University (February 20, 1922);

C. A. FISCHER, Professor of mathematics and astronomy, Trinity College, Hartford, Conn. (December 7, 1922);

ANGELO HALL, Professor of mathematics, U. S. Naval Academy (April 14, 1922);

JAMES MCMAHON, Professor of mathematics, Emeritus, Cornell University (June 1, 1922);

C. M. NOLAND, Professor of mathematics, Howard Payne College, Brownwood, Texas (October 2, 1922);

B. F. SIMONSON, Professor of mathematics, Upper Iowa University (February 27, 1922).

The election of officers for the year 1923 was conducted by mail, and in person at this meeting, as provided in the By-Laws. The tellers reported the result of the balloting as follows:

For President: R. D. Carmichael, 223 votes; H. L. Rietz, 179 votes.

For Vice-Presidents: A. A. Bennett, 173 votes; A. B. Chace, 256 votes; L. P. Eisenhart, 186 votes; B. F. Finkel, 176 votes.

For additional members of the Board of Trustees (to serve until January, 1926): C. F. Gummer, 223 votes; Dunham Jackson, 226 votes; N. J. Lennes, 116 votes; E. H. Moore, 277 votes; R. E. Moritz, 130 votes; Clara E. Smith, 217 votes; Oswald Veblen, 188 votes; J. W. Young, 215 votes.

The following were accordingly declared elected:

President: R. D. Carmichael, University of Illinois.

Vice-Presidents: A. B. Chace, Brown University; L. P. Eisenhart, Princeton University.

Additional members of the Board of Trustees: C. F. Gummer, Queen's University; Dunham Jackson, University of Minnesota; E. H. Moore, University of Chicago; Clara E. Smith, Wellesley College.

The Secretary-Treasurer made his financial report for the year, giving an account of all the business transacted for the Association, up to December 5th, 1922. This report has been audited by a committee consisting of Professors A. D. Pitcher, C. E. Wilder, and H. E. Slaught. The financial report is printed in full below:

Report of the Secretary-Treasurer as Treasurer, Dec. 5, 1922.

Receipts.

Balance Dec. 6, 1921		$ 4,013.98	Sale copies of Monthly	53.00	
1920 indiv. dues	$ 15.00		Sale of old typewriter	15.00	
1920 instit. dues	5.00		Advertising	341.50	
1921 indiv. dues	234.89		Refund from Editor-in-Chief's office	25.00	
1921 instit. dues	38.25		Sale of Register	245.50	
1921 subscriptions	10.50		Interest State Svgs. Bk.	100.12	
1922 indiv. dues	4,822.45		Interest Peoples Bk.	90.39	
1922 instit. dues	586.00		Interest Liberty Bds.	34.38	
1922 subscriptions	734.41				
Contribution to 1922 expenses	413.00		Total 1922 receipts		$ 7,966.39
Initiation fees	202.00				

Expenditures.

Publisher's bills (Oct.–Dec. '21, Feb., Apr., May '22)		$3,169.87	Part expense Register	72.50	
Manager's office		26.53	Printing	272.48	
Editor-in-Chief's office		956.66	Toronto meeting	63.75	
Secretary-Treasurer's office:			Rochester meeting	52.78	
Postage	$205.19		Paid copies of Monthly	24.70	
Bond	5.00		Refund on subscriptions	23.05	
Safety deposit (2 yrs.)	8.00		Library expense	54.90	
Office supplies	28.60		Paid to sections from initiation fee	38.91	
Typewriter	87.64				
Express, tel., etc.	41.23				$1,333.98
Clerical work	355.25				

Total assets to the end of 1922 business	$11,980.37
Total expenditures	5,587.04
Balance to the end of 1922 business	$ 6,393.33
Received on 1923 and later business	406.40
Book balance Dec. 5, 1922	$ 6,799.73

Annals subvention	100.00
Total expenditures	$5,587.04
Cash on hand	34.13
Checking account	372.12
State Savgs. Co. account	2,796.06
Peoples Bkg. Co. account	2,597.42
Liberty Bond	500.00
U. S. Treasury Note	500.00
Bank balance Dec. 5, 1922	$6,799.73

As a separate fund, there was received on the Carus Fund $1,200.00, which, with the interest accruing at 4 per cent., compounded quarterly, amounts at present to $1,236.36

When the accounts were closed on December 5, 1922, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1922 the following items:

BILLS RECEIVABLE.

1922 indiv. dues unpaid	$ 90.00
1922 instit. dues unpaid	42.00
Advertising	230.00
Interest Liberty Bond	10.00
Interest U. S. Treasury Note	11.00
	$383.00

BILLS PAYABLE.

(Either paid or estimated.)

Publisher's bills (7 issues @ $500.00)	$3,500.00
Register	590.00
Manager's office	30.00
Editor-in-Chief's office	150.00
Other editor's postage	20.00
Committee on Membership	55.00
Secretary-treasurer's office	185.00
Annals subvention for 1922	200.00
Init. fees due to sections	100.00
Printing annual ballots, programs, etc.	150.00
	$4,980.00

If to the balance on 1922 business shown in this report, $6,393.33, there be added the amount of bills receivable, $383.00, and there be subtracted the estimated amount of bills payable, $4,980.00, there results an estimated final balance on 1922 business of approximately $1,800.00. The corresponding estimated final balance one year ago on 1921 business was $1,530.00. That a surplus rather than deficit will evidently result from the year's business is due to the generous gift of $400.00 made by Chancellor Chace of Brown University, a contribution much appreciated by the Board of Trustees and by all the members of the Association. Your officers are continually endeavoring to keep the expenses of the Association as low as possible consistent with effective work, and it is their hope that the members will support the Association by continuing their own membership and by urging others to become members.

ANNOUNCEMENT FOR THE ASSOCIATION.

The summer meeting of the Association will be held by invitation at Vassar College on Wednesday afternoon and Thursday, September 5 and 6, 1923, in conjunction with the summer meeting of the American Mathematical Society.

The following persons and institutions, on applications duly certified, have been elected to membership in the Association since the annual meeting:

To individual membership.

J. W. ALEXANDER, Ph.D. (Princeton). Asst. prof., Princeton Univ., Princeton, N. J.
Sr. M. ANGELICA, Ph.D. (Fordham). Head of dept. of math., St. Joseph Coll., Brooklyn, N. Y.
W. L. AYRES. Stud. Instr., Southwestern Univ., Georgetown, Tex.
GLADYS BANES, A.B. (Butler). Instr., Butler Coll., Indianapolis, Ind.
ALLEN BERGER, A.M. (Peabody). Prof., Southeastern State Teachers Coll., Durant, Okla.
HAROLD BLAIR, B.S. (Mich.). Western State Normal Coll., Kalamazoo, Mich.
E. C. BLOM, A.M. (Missouri). Prof., Des Moines Univ., Des Moines, Iowa.
F. A. BRANDNER, B.S. (Kan. State Teachers Coll.). Instr., Ia. State Coll., Ames, Iowa.
MYRTLE C. BROWN, A.M. (Texas). N. Texas State Normal Coll., Denton, Tex.
J. G. BURKE, A.M. (Mt. St. Mary's). Vice-pres., Mt. St. Mary's Coll., Emmitsburg, Md.
Sr. MARY B. CLARKE, A.M. (Cath. Univ. of Amer.). Prof., Loretto Coll., Webster Groves, Mo.
E. McC. CLAYTOR, B.S. (Citadel). Instr., Univ. of the South, Sewanee, Tenn.
R. L. CORNETET, Instr., Boehm Academy, Westerville, Ohio.
C. W. DANCER, A.M. (Ohio St. Univ.). Instr., Univ. of the City of Toledo, Toledo, Ohio.
R. G. DEMAREE, M.S. (Chicago). Prof., Wesleyan Coll., Winchester, Ky.
L. S. DEMAUD, A.B. (Missouri). High School, Oklahoma City, Okla.
WALTER DENSTON, B.A. (Cambridge). Asst. prof., Kenyon Coll., Gambier, O.
H. E. DICKEY, A.B. (Cornell). Prin., High School, Fontanelle, Iowa.
J. S. DONAGHHO, A.M. (Marietta). Prof., Univ. of Hawaii, Honolulu, T. H.
V. H. DOUSHKESS, A.M. (Lafayette). Instr., Lafayette Coll., Easton, Pa.
W. H. DURFEE, M.C.E. (Harvard). Asst. prof., Hobart Coll., Geneva, N. Y.
ANNIE D. DURHAM, A.B. (Tex. Presby. Coll.). Instr., Texas Presbyterian College, Milford, Tex.
J. M. EARL, A.B. (Carleton). Instr., Mich. Agri. Coll., E. Lansing, Mich.
R. M. ELLIOTT, B.S. (Pacific Coll.). Instr., Univ. of Ore., Eugene, Ore.
HAL FOX, B.S. (Miss. A. & M. Coll.). Asso. prof., Miss. A. & M. Coll., Agricultural College, Miss.
O. A. GEORGE, A.M. (Minn.). Head of dept. of math., High School and Jr. Coll., Mason City, Iowa.
L. O. GHORMLEY, Ph.B. (Wooster). Instr., Univ. of Tenn., Knoxville, Tenn.
MILDRED A. GIESECKE, B.S. (Northwestern). Chicago, Ill.
A. E. GANT, M.S. (Chicago). Instr., Bradley Poly. Inst., Peoria, Ill.
H. E. H. GREENLEAF, B.S. (Boston). Depauw Univ., Greencastle, Ind.
J. M. GUILLIAMS, A.B. (Central Nor. Coll.). Teacher, Berea Normal School, Berea, Ky.
I. J. GWINN, A.B. (Morningside). Instr., Morningside Coll., Sioux City, Ia.
HILLEL HALPERIN, A.M. (Columbia). Asso. prof., Texas A. & M. Coll., College Station, Texas.
EMMA E. HANTHORN, A.B. (Nebr.). Teacher, State Teachers Coll., Kearney, Nebr.
BERTHA I. HART, A.B. (W. Md. Coll.). Instr., Western Md. Coll., Westminster, Maryland.
R. N. HASKELL, B.S. (Chicago). Instr., Mich. Agric. Coll., E. Lansing, Mich.
D. A. HATCH, Eng. of Mines (Lafayette). Asst. prof., Lafayette Coll., Easton, Penn.
JULIA L. HAWKINS, B.S. (Chicago). Instr., Okla. Coll. for Women, Chickasha, Okla.
JEANETTE HENNA, A.B. (Westhampton). Teacher, Marion Coll., Marion, Va.
C. E. HILLE, Ph.D. (Stockholm). Instr., Princeton Univ., Princeton, N. J.
G. G. HUBBARD, A.B. (Oberlin). Asst. prof., Colo. School of Mines, Golden, Colo.
DANIEL HULL, M.S. (Notre Dame). Prof., Univ. of Notre Dame, Notre Dame, Ind.
H. E. HUNTER, M.S. (Illinois). Asst. prof., State Man. Training School, Pittsburg, Kans.
P. M. ILOFF, A.M. (Mich.). Instr., State Teachers Coll., Chico, Calif.
FRANCES H. JACKSON, A.M. (Columbia). Head of dept. of math., Salem Coll., Winston-Salem, N. C.

W. C. JANES, A.M. (Nebr.). Instr., Kans. St. Agri. Coll., Manhattan, Kan.
ABIGAIL E. JOHNSON, B.S. (Columbia). Head of dept. of math., H. S., Morristown, N. J.
F. E. JOHNSTON, A.B. (George Washington). Asst., Univ. of Ill., Urbana, Ill.
KATHRYN M. KENNEDY, A.B. (Ind. State Normal). Normal Training School, Terre Haute, Ind.
C. D. KILLEBREW, M.S. (Ala. Poly. Inst.). Prof., Ala. Poly. Inst., Auburn, Ala.
D. O. KIME, M.S. (Univ. of Ill.). Western Union Coll., Le Mars, Ia.
E. B. KIRK, B.S. (Louisville). Grad. stud., Harvard Univ., Cambridge, Mass.
M. M. LEVITA, A.M. (Univ. of Penn.). Instr., Temple Univ., Phila., Penn.
W. F. LONG, A.B. (Franklin & Marshall). Prof., Franklin and Marshall Coll., Lancaster, Penn.
C. A. LOVELL. Instr., Mississippi Coll., Clinton, Miss.
G. A. LYLE, B.S. (Hampden-Sidney). Instr., Lehigh Univ., Bethlehem, Penn.
A. C. MADDOX, A.M. (Columbia). Prof., State Normal Coll., Natchitoches, La.
Sr. B. M. MASKELL, A.B. (Coll. of St. Elizabeth). Teacher, Coll. of St. Elizabeth, Convent Station, N. J.
T. E. MASON, Ph.D. (Indiana). Asso. prof., Purdue Univ., LaFayette, Ind.
ELLEN M. MCMICHAEL, A.B. (Rockford). Head of dept. of math., Blackburn Coll., Carlinville, Ill.
Sr. MERCEDES, A.B. (St. John's). Teacher, Central Catholic High School, Toledo, Ohio.
J. S. MORREL, A.B. (Southwestern). Instr., Univ. of Tenn., Knoxville, Tenn.
F. H. MURRAY, A.M. (Harvard). Instr., Univ. of W. Va., Morgantown, W. Va.
EMMA J. OLSON, A.B. (S. Dak.). Asst. prof., Tex. State Coll. for Women, Denton, Tex.
P. C. OVERSTREET, A.B. (Asbury). Central Holiness Univ., University Park, Ia.
L. M. PASSANO, A.B. (Johns Hopkins). Asso. prof., Mass. Inst. Tech., Cambridge, Mass.
EDITH M. PERRY, A.M. (Okla.). Asst., Extension Dept., Univ. of Okla., Norman, Okla.
O. J. PETERSON, A.M. (Michigan). Instr., Univ. of Mich., Ann Arbor, Mich.
JESSE PIERCE, B.S. (Idaho). Asso. prof., James Millikin Univ., Decatur, Ill.
MARY G. POPE, L.I. (Winthrop). Instr., Winthrop Coll., Rock Hill, S. C.
A. G. RAU, Ph.D. (Moravian). Prof., Moravian Coll., Bethlehem, Penn.
MINNA E. REESE, B.S. (Northwestern). Hubbard, Iowa.
LENA E. REYNOLDS, A.M. (Calif.). Head of dept. of math., H. S. & Jr. Coll., Fullerton, Calif.
L. V. ROBINSON, A.M. (Texas). Head of dept. of math., Meridian Coll., Meridian, Tex.
D. S. SAUND, A.M. (Calif.). Berkeley, Calif.
H. L. SHIRLEY, A.B. (Indiana). Instr., Univ. of Nevada, Reno, Nev.
C. S. SHIVELY, Ph.D. (Denver). Prof. Juniata Coll., Huntingdon, Penn.
JOSEPH SPEAR, A.B. (Harvard). Prof., Northeastern Univ., Boston, Mass.
D. V. STEED, Ph.D. (Calif.). Asst. prof., Univ. of South. Calif., Los Angeles, Calif.
C. G. T. STIPE, A.B. (Michigan). Instr., Univ. of S. Dak., Vermillion, S. Dak.
C. E. STOUT, B.S. (Heidelberg). Instr., Heidelberg Univ., Tiffin, O.
EDITH M. SWANSON, A.B. (Minn.). Instr., Univ. H. S., Vermillion, S. Dak.
CARRIE B. TALIAFERRO, B.S. (Columbia). Asso., State Normal School, Farmville, Va.
H. W. TAYLOR, A.M. (Kansas). Dept. of phys., Coll. of Emporia, Emporia, Kan.
Rev. JOHN THEOBALD, A.B. (Columbia Coll.). Columbia Coll., Dubuque, Ia.
W. M. THORNTON, LL.D. (Hampden-Sidney). Dean of engg., Univ. of Va., University, Va.
J. T. VALLANDINGHAM, A.B. (Georgetown). Prof., Cumberland Jr. Coll., Williamsburg, Ky.
C. A. VANVELZER, Ph.D. (Hillsdale). Prof., Carthage Coll., Carthage, Ill.
MARTHA W. WATT, A.M. (Columbia). Asst. prof., Wheaton Coll., Norton, Mass.
E. A. WHITMAN, A.M. (Pittsburgh). Instr., Carnegie Inst. Tech., Pittsburgh, Penn.
C. S. WHITNEY, A.B. (Okla.). Grad. stud., Univ. of Okla., Norman, Okla.
W. E. WILBUR, M.S. (Maine). Instr., New Hampshire Coll., Durham, N. H.
W. P. YANCEY, A.M. (Woodstock). Prof., St. Ambrose Coll., Davenport, Ia.
P. R. YODER, A.M. (Univ. of Kan.). Prof., Blue Ridge Coll., New Windsor, Md.

To institutional membership.

CALIFORNIA INSTITUTE OF TECHNOLOGY, Pasadena, Calif. Prof. P. S. Epstein, Rep.
COLUMBIA COLLEGE, Dubuque, Ia. Rev. E. D. Howard, Rep.
LA. STATE NORMAL COLLEGE, Natchitoches, La. Prof. A. C. Maddox, Rep.
ST. VIATOR COLLEGE, Bourbonnais, Ill. Prof. J. R. Plante, Rep.

W. D. CAIRNS, *Secretary-Treasurer.*

NINTH REGULAR MEETING OF THE KANSAS SECTION.

The ninth regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, January 20, 1923, in connection with a meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, the first of which was a joint session with the Kansas Association. Professor W. H. Garrett presided at the first session and Professor A. E. White at the second.

The attendance was sixty-two, including the following twenty-three members of the Association:

Florence Black, W. H. Garrett, W. A. Harshbarger, T. B. Henry, Emma Hyde, S. Lefschetz, C. F. Lewis, T. Lindquist, O. B. Loewen, Anna Marm, U. G. Mitchell, Thirza Mossman, P. Pretz (institutional representative), B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, H. G. Titt, Eula Weeks, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor LINDQUIST; Vice-Chairman, Professor TITT; Secretary-Treasurer, Professor MITCHELL.

The following papers were presented:

(1) "The development of the junior high school movement in Kansas and its effect on the efficiency of mathematics instruction in the seventh, eighth and ninth grades" by Professor T. LINDQUIST.

(2) "The National Committee's report on the reorganization of secondary mathematics" by Dr. EULA A. WEEKS.

(3) "Some peculiar limiting functions and their graphs" by Professor G. W. SMITH.

(4) "The teaching of unified mathematics" by Professor P. PRETZ.

(5) "The area of a cone having an elliptical base" by Miss THIRZA MOSSMAN.

(6) "A new method of determining sufficient conditions for real roots of equations" by Miss WEALTHY BABCOCK, (by invitation).

(7) "A map of $\sinh z$" by Professor T. B. HENRY.

At the joint session there was a general discussion of the topic "Should the State Board of Education recognize the existence of the junior high school system in the state?" and it was voted to appoint a committee of three to work for the standardization of junior high school mathematics in the state.

After the presentation of each paper there was a general discussion. Abstracts of the papers are given below, the numbers corresponding to numbers in the list of titles:

1. Professor Lindquist brought out the following facts: (*a*) the first junior high schools in Kansas were organized in 1911; (*b*) of 61 cities with population above 2000 which replied to a questionnaire in 1921, 22 reported using the 6–3–3 plan, 21 the 6–2–4 plan and 18 no junior high school organization; (*c*) schools using the 6–3–3 plan reported 9200 junior high school pupils and those using the 6–2–4 reported 5000; (*d*) this indicates that 64 per cent. of the junior high school pupils in Kansas are in schools which have a three-year junior high school course.

Since a number of cities known to have junior high schools failed to answer the questionnaire, 20,000 is a conservative estimate of the number of pupils attending junior high schools in Kansas in 1921; (*e*) nearly all of the teachers in the junior high schools of the state have had at least two years of college training and about one half hold bachelor's degrees.

2. Dr. Weeks gave a general résumé of the work of the National Committee and emphasized especially the need for reading, discussion and criticism by teachers of the work of the Committee, if its reports are to be most valuable.

3. Professor Smith showed that expressions can be set up, the limit of which represents a certain function $f(x)$ for $0 \leqq x < a$ and another function $g(x)$ for $x > a$. By properly introducing $\sqrt{x^2 - b^2}$ and $\sqrt{c^2 - x^2}$ he set up expressions which have real values only in the interval $b \leqq x \leqq c$. Many examples were given and for some of these several of the approximation curves were plotted.

4. Professor Pretz stated that the teaching of unified mathematics is an issue in the field of education that must be met and judged on its merits by the teachers of mathematics themselves. On the basis of his own experience and the experience and opinions of others he believed that unified courses can be taught successfully in both secondary schools and colleges. He advocated the teaching of unified mathematics, chiefly because he believed the unified scheme to be more broadly scientific than other plans. The proper preparation of teachers of unified courses was emphasized.

5. The lateral area of a cone whose base is an ellipse can be expressed as an elliptic integral. If the vertex of the cone is over the principal axis of the ellipse, the area is computable. This area is in general represented by an elliptic integral, but in the special cases of the right circular cone, the cone of zero altitude, and the cone whose vertex lies on a certain hyperbola in a plane vertical to the base, the area integral is non-elliptic.

6. Miss Babcock outlined a method of determining conditions to be placed upon the coefficients of an equation such that the roots of the equation would be real. The general symmetric determinant of order equal to the degree of the equation was expressed in terms of its coaxial minors of the first and second orders, and these minors then expressed in terms of the coefficients of the equation, so that, as a result, the equation was expressed in the form of a symmetric determinant. Conditions to be placed upon the coefficients of the equation, sufficient to make the roots real, may be determined from this symmetrical determinant form of expression. These conditions were given for equations of the third and fourth degrees.

7. Professor Henry presented a comprehensive discussion of the conformal representation of hyperbolic functions of a complex variable, with graphic representations of the results obtained.

U. G. MITCHELL, *Secretary-Treasurer.*

HISTORICAL-MATHEMATICAL PARIS.

By DAVID EUGENE SMITH, Columbia University.

I. Ile de la Cité and the Voltaire-Châtelet Paris.

The World War has naturally turned the steps of many of our advanced students to the paths their intellectual ancestors trod soon after the American and French revolutions, namely, to Paris. There will still be large numbers who go to Germany and England, and many who go to Italy, but for some years to come it is probable that Paris will attract American students more than it ever has in the past and more than any other single city of Europe. For these students, and for the more casual visitor of mathematical tastes as well, this article has been prepared in the hope that new interest may be added to their sojourn in what is, all things considered, the most attractive city of the world. Having spent much time there during repeated visits spread over a period of more than forty years, I have naturally come to know a considerable number of the places of mathematical interest, and my collection of autograph letters of those who have made the science what it has become in the last three or four centuries has supplied considerable information as to where the various writers and their correspondents lived and labored and died. I have also been aided by such works as those of the learned M. Cain (for example, his *Promenades dans Paris*) and by the more detailed but less well-written work of the Marquis de Rochegude (*Promenades dans toutes les Rues de Paris*), but I have naturally selected only a few of the many spots of historic interest that anyone could readily find if he should attempt such a pretentious piece of work as a book upon the subject. In many cases the houses mentioned are still standing, as, for example, two on the Rue de Bac and one on the Rue de Lille, but in any event the mere location has enough interest to make some reference to it worth while.

Beginning with the most ancient part of the city,—the Lutetia of Cæsar's time, now a part of the Ile de la Cité, upon which Notre-Dame stands,—we may turn to that little gem of Gothic architecture, the Sainte-Chapelle in the ancient Palais de Justice. It was constructed in 1245–1248 by Saint Louis as a fitting receptacle for the Crown of Thorns and a portion of the True Cross, and it seems to have had as one of its canons Rollandus, whose general treatise on mathematics, written *c.* 1425, has come down to us in manuscript copy.[1]

On the north side of the Palais de Justice, and entered from the Quai de l'Horloge, is the Conciergerie, the most ancient prison of the city. It was here that Jean Silvain Bailly[2] was confined before his execution in the Reign of Terror. While he was still in favor of the revolutionists he was mayor of Paris, although carrying on his studies in mathematical astronomy, and lived in the Hôtel de la Mairie which stood where is now No. 14 of the Rue des Capucines, a street which runs from the Rue de la Paix to the Boulevard des Capucines. He also lived for a time at No. 21 (old numbering) of the Rue de Chaillot, once the principal street

[1] This copy is now in the library of Mr. George A. Plimpton, New York City.

[2] Born in Paris, Sept. 15, 1736; guillotined Nov. 12, 1793.

of the village of Chaillot, the Colloelum of the 11th century, but now a residential section southeast of the Arc de Triomphe de l'Étoile, so called to distinguish it from the Arc de Triomphe du Carrousel. It was on July 17, 1791, that he and Lafayette directed the charge on a mob which had demanded the surrender of the king. After his execution (1793) in the Place de la Concorde[1] his body was buried in the ancient cemetery (1659–1865) of the Madeleine, now (since 1865) the Square Louis-XVI, on the Rue Pasquier, a little south of the Gare Saint-Lazare.

If we rank Voltaire[2] in our guild because of his work on the philosophy of Newton,[3] we shall naturally find many spots in Paris connected with his name, and portraits and statues in great number and often of much excellence. The present Rue Molière, running from the Avenue de l'Opéra to the Rue Richelieu, for example, was once the Rue Traversière, and at the old number 25 was a house which was rented to the Marquise du Châtelet,[4] and there Voltaire lived for some time, setting up a little theatre for his plays. Around the corner, at No. 8 of the Rue de Richelieu, the street on which the Bibliothèque Nationale fronts, was the café of Charlotte Bourette, who was known as the Muse Limonadière, and whom Voltaire esteemed for her wit. Farther up the Rue de Richelieu, at No. 102, stood a house which Voltaire owned and in which his niece, Mme. Denis, lived after the death of the Marquise du Châtelet. Next door, at No. 100, stood the house of Voltaire's friend, Mme. de St. Julien, whom he often visited. Voltaire also lived (1732 and 1733) at what is now No. 20, Rue de Valois, in the same vicinity, east of the Palais Royal. Not far from here, at No. 161, Rue Saint-Honoré, is the Café de la Régence, which I well recall as still prominent in the artistic life of Paris when I was a boy. Its predecessor stood a little to the east, at the Place du Palais Royal, and was frequented by Voltaire as well as by Benjamin Franklin, Diderot, Napoleon, and other makers of history. Over on the Ile Saint-Louis, at No. 2, on the Rue Saint-Louis-en-l'Ile, is the hôtel (mansion) of Nicolas Lambert de Thorigny, sometime president of the Cour des Comptes, built in 1680. The Marquise du Châtelet lived there for a time, and Voltaire was, as usual, a guest of the house. His sister, Mme. Mignot, mother of Mme. Denis (to whom Voltaire was greatly attached), lived at No. 133, Rue Saint-Antoine, a continuation of the Rue de Rivoli and leading into the Place de la Bastille. Although the Bastille has long since ceased to exist,[5] when the wanderer stands upon its ancient site he may reflect that Voltaire was twice imprisoned there,[6] for his rash utterances on the rights of man. Voltaire was baptized (1694) in the church of Saint-André-des-Arcs, which was built in 1210.

[1] The guillotine was at the entrance to the Champs Elysées.

[2] François Marie Arouet, who took the name of Voltaire (anagram on Arouet le jeune = Arovet l. i.); born at Paris, November 21, 1694; died at Paris, May 30, 1778.

[3] *Élémens de la philosophie de Neuton*, Amsterdam, 1738. Compare this MONTHLY, *1921*, 303–305.

[4] Gabrielle Émilie Le Tonnelier de Breteuil, born at Paris, December 17, 1706; died at Commercy, September 10, 1749. She spelled her married name in the full form,—Chastelet.

[5] The upper part of the Pont de la Concorde was built from the stones of the Bastille.

[6] In 1717 and 1726, more than sixty years before its destruction.

It stood on the present Place Saint-André-des-Arts,[1] near the Point Saint-Michel, and was demolished about 1800. In 1793 it became the Temple de la Révolution. Voltaire once worked as a clerk in the office of Maître Alain, No. 1, Rue des Grands-Degrés, so called from the steps leading down to the quai, and he became a mason in the lodge of the Neuf Soeurs which stood at No. 80 of the Rue Bonaparte; but the atmosphere of the Quartier Latin was perhaps not so well suited to his maturer years, although he lived for a time in Rue Mazarine and in 1743 was living at No. 23, Rue Fontaine Molière. He died in the house of the Marquise de Villette, at No. 27, Quai Voltaire, as an inscription states. The present name of the quai, formerly the Quai des Théatins, was given in memory of this event, as was that of the Rue Voltaire which branches off at No. 211. His final resting-place is appropriately in the Panthéon, the Valhalla of France.

As to busts, bas-reliefs, and statues of Voltaire, Paris has been over-generous. Houdon's bust in the Comédie Française is the best known, but the statue by Caillé (1885) on the Quai Malaquais is also familiar to every visitor to the book-stalls on the Rive Gauche.

As to the Marquise du Châtelet[2] and her family, Tonnelier de Breteuil, there are various interesting spots connected with each. The family owned a hôtel at No. 14, Rue Portefoin, a little to the southeast of the Conservatoire des Arts et Métiers. They also owned (1760) a place at No. 56, Rue des Francs-Bourgeois, near the Palais des Archives Nationales, and somewhat earlier (1728) one at No. 4, Place des Vosges, on the same street. In 1752 the marquise was living at No. 18 of the same Place.

II. The Quartier Latin.

Returning in our wanderings to the Quartier Latin, and to names more mathematical, at No. 1, Rue de la Sorbonne, Hermite[3] died in 1901, and on the walls of the Église de la Sorbonne is his *médaillon*. At No. 2 of the Rue Rollin, which opens on the west side of Rue Monge, Pascal[4] died, as an inscription states, at the house of his sister, Marguerite Périer, who afterward wrote his biography. Descartes lived at No. 14 of the same street. A little to the south, parallel to and east of Rue Monge, is the Rue de la Clef where, at No. 38, Monge[5] lived for a time. He also lived in Rue de Dragon, a little to the west of Saint-Germain-des-Prés and to the south of Boulevard Saint-Germain. One of the letters[6] written from Linz, when he was with Napoleon on the Austrian campaign, is addressed to Madame Monge at "Rue neuve Belle chasse No. 3," the Rue de Bellechasse (formerly Belle Chasse). The part between the Rue Saint-Domi-

[1] The change from *arcs* to *arts* is relatively recent.

[2] Compare this Monthly, *1921*, 368–369.

[3] Charles Hermite (1822–1901), who proved the transcendence of e.

[4] Blaise Pascal, born at Clermont-Ferrand in 1623; died at Paris in 1662.

[5] Gaspard Monge, born at Beaune in 1746; died at Paris in 1818. He is known chiefly for his work in descriptive geometry.

[6] Whenever such letters are mentioned it is to be understood that they are at present in the author's collection.

nique and the Rue de Grenelle was formerly called Rue Neuve Belle Chasse. It seems, from the old maps, that this house stood at the corner of the present Rue Saint-Dominique. Monge also lived for a time at Nos. 7–9, Rue Bonaparte, as stated later.

To the west of Rue Monge, and nearly parallel with it, is the Rue d'Ulm which runs from the Panthéon south to Rue Claude Bernard. At No. 43, Rue d'Ulm, is the École Normale Supérieure, founded in 1795 and occupying the present building since 1847,—an institution with which have been connected many mathematicians of prominence, Jules Tannery being one of the last of those who have now passed away. At the next corner to the north and east of this school the Rue Lhomond continues the line of the Rue des Fosses Saint-Jacques. It was formerly called the Rue des Postes, and a letter was written from this street by Pierre Bouguer[1] on October 31, 1750. The first street to the north is now known as the Rue de l'Estrapade, known a century ago as the Rue de la Vieille-Estrapade, so called from the *estrapado* or *strappado* punishment there inflicted upon soldiers in early times. At No. 11, in a mansion still standing, there lodged Georges Marie Raymond,[2] professor of mathematics in Geneva, whose contributions to algebra appeared frequently in Gergonne's *Annales*.

At No. 9 of the Quai Malaquais, which extends westward from the Palais de l'Institut, Legendre[3] lived for four years (1809–1813). The building stands next to the École des Beaux Arts, on what was formerly part of the grounds of the ancient Abbey of Saint-Germain-des-Prés. Before this he lived at No. 12, Rue Condé, a street running north from the Palais du Luxembourg, one of his letters having been written from there in 1804.

Near the Pont Saint-Michel, and west of the Boulevard, is the Rue Saint-André-des-Arts, dating from 1179,—formerly Saint-André-des-Arcs. At No. 52 of this street, a fine old mansion at the corner of Rue des Grands Augustins, Joseph-Louis-François Bertrand[4] was born in 1822. The next parallel street to the north is Rue Christine where, in a dignified old mansion at No. 2, Laplace[5] lived in 1802. It leads to the east into the narrow Rue de Savoie, with numerous old houses, where, at No. 13 as an inscription states, Sophie Germain[6] died on June 27, 1831.

The Rue Mazarine runs southward from the Institut, and leads into the Rue de l'Ancienne Comédie where, at the oldest café in Paris, the Procope (No. 13), founded in 1689, d'Alembert, Voltaire, and many others among the intellectuals gathered. The café still stands, shorn of its ancient prestige, as is also the case with the old Comédie Française across the street, at No. 14.

[1] Born in 1698; died at Paris in 1758. He was engaged with Condamine and others on the figure of the earth. He contributed to the geometry of curvilinear figures.

[2] Born in 1769; died in 1839. Delambre wrote a letter to him at this address in January, 1822.

[3] Adrien-Marie Legendre, born at Toulouse in 1752; died at Paris in 1833.

[4] He was professor of mathematical physics in the Collège de France and died in 1900.

[5] Pierre Simon Laplace, born in 1749; died in 1827. Known primarily for his work in celestial mechanics.

[6] She was born in 1776. She is known for her work on the theory of elastic surfaces.

Puissant [1] lived in the Rue Mazarine in 1826, as is shown by a letter written by Bouvard [2] to him in 1826, and Gaston Darboux[3] lived at No. 3 of the same street,—his official residence as secrétaire perpétuel de l'Académie des Sciences. Where the Rue Mazarine becomes the Rue de l'Ancienne Comédie the Rue de Buci runs to the west, and at No. 19 Mme. Denis, niece of Voltaire, was joint proprietor of a residence. In the Rue de Seine, nearly parallel to the Rue Mazarine and the best place in Paris for the collector of early mathematical portraits, Legendre lived at one time and the widow of the unfortunate Condorcet [4] died in 1822. She had lived before that (1812) in the Rue de Penthière, north of the Champs Elysées. At No. 4 of the southern part of Rue Mazarine, there called the Rue de Tournon, J. L. F. Bertrand died in 1900, and at No. 12, in an elaborate old mansion near the entrance to the Palais du Luxembourg, Cauchy [5] and Leverrier [6] both lived.

Not far to the east of the Rue de Tournon is the Place de l'Odéon where the Café Voltaire was much frequented by Voltaire and his friends. From here to the Boulevard Saint-Michel there runs the Rue Racine in which, at No. 30 as an inscription states, there died Auguste Comte (1798–1857), the founder of positivism, a writer on the philosophy of mathematics, and the editor of some of Pascal's works.

In Rue Bonaparte, the next street to the west of the Rue de Seine, Monge [7] lived in the hôtel of the Marquis de Persen (Nos. 7–9),—a building now given over to commercial uses. To the west of this street, and extending to the south from the church of Saint Sulpice, at No. 15 of the narrow Rue Servandoni with its ancient structures, the widow of François Vernet, the sculptor,[8] concealed Condorcet while he was writing his *Esquisse du progrès de l'esprit humain* (1793), the last work undertaken by him before he poisoned himself to escape the guillotine. The old house still stands, bearing a proud inscription of the incident, but looking dilapidated enough to conceal with perfect safety any unfortunate seeker after oblivion. Just before this, Condorcet lived a short distance away, at No. 71, Rue de Lille,[9]—from 1640 to 1792 known as the Rue de Bourbon, and again by the same name from 1814 to 1830. Twenty years before this time Monge wrote a letter to Condorcet, addressing it to him "Chez M^{r} le M^{is} d'Essé, Rue de Bourbon, St. germain," but no number was given. It was then one of the most fashionable parts of Paris and, being near the Quai d'Orsay, is still a diplomatic center. The well-known Rue du Bac crosses this street, and at No. 26 Baron

[1] Louis Puissant, born in 1769; died at Paris in 1843. He wrote extensively on algebra, geometry, and geodesy.

[2] Alexis Bouvard (1767–1843) the astronomer.

[3] Born at Nîmes, 1842; died at Paris, February 23, 1917.

[4] Marie-Jean-Antoine-Nicolas Caritat, Marquis de Condorcet, born at Nemours in 1730; died near Paris in 1793. He poisoned himself to escape the guillotine.

[5] Augustin-Louis Cauchy, born at Metz in 1788; died at Paris in 1867.

[6] Urbain-Jean-Joseph Leverrier (Le Verrier, 1811–1877) lived there in 1853.

[7] See page 109.

[8] He died in 1784.

[9] The fine mansion may be seen by entering the court.

Charles Dupin[1] died in 1873. The house still stands, and at No. 108, as an inscription states, Laplace died in 1827. It also crosses, farther south, the Rue de Sèvres where, in a mansion formerly standing at No. 16, Madame de Récamier spent the last thirty years of her life and held a notable salon at which all the academicians were received. Among those who frequented her house were Arago and Ampère, and it is of some interest to know that Victor Hugo was baptized there. The building was the ancient Abbaye aux Bois and was demolished in 1908. A few minutes' walk to the southwest of this locality, and near the Gare Montparnasse, is the modern Rue Littré where, at No. 5, Émile Lemoine[2] (1840–1912) lived.

Few who visit this scholastic and artistic part of Paris fail to enter the Musée de Cluny, but probably not many of these recall the fact that the old palace which houses it has a mathematical as well as an archeological interest. Lalande (1732–1807),[3] however, lived there for a time, and Charles Messier (1730–1817), the astronomer, followed him and died there in 1817, the tower serving at that time as the Marine Observatory. It was not until 1833 that M. Du Sommerard (1779–1842) installed his great collection there, the government acquiring it ten years later.

The Panthéon, too, is not usually connected in thought with mathematics, for the inscription, "Aux grands hommes, la Patrie reconnaissante," is not ordinarily associated in the public mind with the mathematicians of the country. Nevertheless there is an indirect interest in recalling that it was here, in February 1851, that Foucault made his experiments relating to the pendulum and the rotation of the earth. He is not buried here, however, but in the cemetery of Montmartre.

Coming out from the Panthéon we face the Rue Soufflot. At No. 14 there stood, from 1217 to 1790, the convent of the Dominicans, or Jacobins, where Albertus Magnus[4] taught in the Middle Ages. Around to the right, as one leaves the Panthéon, stands the Église Saint-Étienne-du-Mont, dating from the 16th and 17th centuries, and it is here that Pascal was buried, as is recorded by an inscription on a pillar back of the chancel.[5] Descartes[6] was also buried here, but his remains were later (1819) transferred to Saint-Germain-des-Prés. In the Bibliothèque Saint-Geneviève nearby is the oldest known French algorism, a parchment manuscript of *c.* 1275, but of course the great collection of mathematical manuscripts is in the Bibliothèque Nationale.

The Sorbonne lies a little to the northeast of the library of Saint-Geneviève,—not the ancient building which I remember as a boy, and where generations of

[1] Born in 1784; died in 1873. Prominent because of his works on mechanics and differential geometry. See this MONTHLY, *1921*, 121.

[2] One of his letters is dated there October 13, 1894.

[3] See this MONTHLY, *1921*, 207.

[4] Count of Bollstädt and bishop of Regensburg. His works include a certain amount of astronomy and some mention of Pythagorean arithmetic. Saint Thomas Aquinas also taught there.

[5] See this MONTHLY, *1921*, 64.

[6] Born at La Haye, March 31, 1596; died at Stockholm, February 11, 1650.

mathematicians had taught since the Middle Ages, but the new and imposing edifice which has gradually replaced its ancestor during the last forty years. Across the Rue Saint-Jacques, to the east, is the Collège de France, where many of the greatest of the French mathematicians have taught. The present building was completed about 150 years ago. The square in front was formerly called the Place Cambray, and a letter of Lalande was written there on October 14, 1800, the year after the death of Montucla whose history of mathematics he edited in its second edition, the letter having probably been written at the collège.

Three letters of Jean-Nicolas-Pierre Hachette[1] were written from "Rue d'Enfer St Michel n° 31." The Boulevard Saint-Michel was opened during the Second Empire, and its present name dates only from 1867. It was formed by straightening and enlarging the ancient Rue de la Harpe, Rue d'Enfer, and other minor streets. The Rue d'Enfer was the portion lying south of the present corner of the Rue Soufflot. In late Roman times it was the Via Infera (voie inférieure, the lower route), a name corrupted as early as the 13th century to Rue d'Enfer (hell's street). The next street to the south of Rue Soufflot is Rue Royer-Collard, known before 1846 as Rue Saint-Dominique-d'Enfer. At No. 15 is found the Impasse Royer-Collard (opened in 1590), formerly called the Impasse Saint-Dominique. It was here that Hachette probably lived, for another of the letters bears the address "Impasse st. dominique d'enfer." Poisson[2] dates a letter from No. 20 of the same street, where he seems to have been living in 1814.

When Rue Soufflot was completed, in the middle of the 19th century, Rue Hiacinte (Hyacinthe) was closed. It ran from the eastern entrance of the Luxembourg gardens to the present Rue Saint-Jacques, and Hachette also lived here,—at No. 20 in 1814 and at No. 8 in 1830, as is shown by two other letters of his.

Down by the river the Palais de l'Institut, built in 1663, is of course closely connected with the history of mathematics, especially in the class of the Académie devoted to this subject and physics. D'Alembert,[3] for example, was the secrétaire perpétuel during the latter part of his life.

(*To be concluded in the next issue.*)

CYCLIC OPERATIONS ON DETERMINANTS.

By A. L. CANDY, University of Nebraska.

"A determinant is not altered in value by adding to all the elements of any column (or row) the same multiples of the corresponding elements of any number of other columns (or rows)."

This is a well-known theorem that is stated in substantially the same form in

[1] Jean-Nicolas-Hachette (1769–1834), well known for his works on algebra and geometry.

[2] Siméon-Denis, Baron Poisson (1781–1840). He wrote on probability, equations, and the calculus.

[3] Jean-Baptiste-le-Rond d'Alembert, born at Paris in 1717; died at Paris in 1783.

all textbooks on determinants. Furthermore, the determinant is not altered in value if more than one column (or row) be operated upon in this manner.

As far as I have examined the textbooks no author has given an explicit statement of this extension, although all make use of it in their illustrative examples. But what is of more importance, I believe no one has hitherto pointed out any exception to this general extension of the above theorem. There is, however, a very important exception,[1] some of the results of which I shall try to show. This exception may be stated as follows:

A cyclic operation upon the rows (or columns) of a determinant will change the value of the determinant, if the cycle is complete.

Let Δ_s represent the determinant formed by adding the rows of Δ cyclically s in a set. When necessary to distinguish the order n of Δ we may write ${}_n\Delta$ and ${}_n\Delta_s$.

For example, let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then by definition

$$\Delta_2 = \begin{vmatrix} a_1 + a_2, & b_1 + b_2, & c_1 + c_2 \\ a_2 + a_3, & b_2 + b_3, & c_2 + c_3 \\ a_3 + a_1, & b_3 + b_1, & c_3 + c_1 \end{vmatrix}.$$

Taking the sum of all the rows for a new first row, and taking out the factor 2, gives a determinant which can be easily reduced to Δ itself. Therefore, ${}_3\Delta_2 = 2 \cdot {}_3\Delta$. When $n = 4$ we have the determinant ${}_4\Delta_2$ equal to zero, since the sum of its first and third rows is the same as the sum of its second and fourth rows. In like manner it can be shown that for any value of n, ${}_n\Delta_2 = 0$ or $2_n\Delta$ according as n is even or odd.

A determinant of the form

$$\begin{vmatrix} k_1 & k_2 & k_3 & k_4 \\ k_4 & k_1 & k_2 & k_3 \\ k_3 & k_4 & k_1 & k_2 \\ k_2 & k_3 & k_4 & k_1 \end{vmatrix}$$

is called a *circulant*. The elements of the first row may have any values whatsoever, and a particular circulant will be completely known when the elements of the first row are given.

Let C_s represent a circulant in which the first s elements of the first row are 1, and the others all 0; when it is necessary to distinguish the order n of the circulant we shall write ${}_nC_s$.

THEOREM I. *For all values of n and s, $n > s$, ${}_n\Delta_s = {}_nC_s \cdot {}_n\Delta$.*

This appears at once if we form the product by taking ${}_nC_s$ by rows, and ${}_n\Delta$ by columns.

For example, when $n = 3$, we have

$${}_3C_2 \cdot {}_3\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = {}_3\Delta_2.$$

[1] This "exception" covers an error that is frequently made by students.

Hence, in order to determine the effect of adding the rows of ${}_n\Delta$ cyclically s in a set, it will be sufficient to evaluate ${}_nC_s$. For this purpose we shall prove the next three theorems.

THEOREM II. *If n and s have a common factor, ${}_nC_s = 0$.*

Let $n = km$ and $s = kr$, where m and r are prime to each other. In any column (as well as in any row) the elements can be grouped into m sets, k in a set. Of these there are r consecutive sets in which the elements are all 1's, and $(m - r)$ consecutive sets in which they are all zeros. Now any series of m rows, whose numbers are $i, i + k, i + 2k, \cdots, i + (m - 1)k$, where $i \not\equiv k$, includes in every column one element, and only one, from each of these sets, that is, r 1's and $m - r$ zeros. Hence, if we replace any row of such a series by their sum, we shall get a new row in which each element is r. Since in this way we can make k rows alike, the determinant must be zero.

THEOREM III. $$\frac{{}_nC_{n-s}}{n-s} = \frac{{}_nC_s}{s}.$$

In ${}_nC_s$ take the sum of all the rows for a new first row, and take out the factor s. Then subtract the first row from each of the others, and change the signs in all these $n - 1$ rows. Then permute the rows cyclically until the last s rows become the first s rows. These operations will change the sign of the determinant $n - 1 + s(n - 1) = (n - 1)(s + 1)$ times. We now have

$$ {}_nC_s = (-1)^{(n-1)(s+1)} \cdot s\Delta_1,$$

where Δ_1 is precisely the same determinant as that obtained from ${}_nC_{n-s}$ by adding to the row whose number is $s + 1$ all the other rows, and taking out the factor $n - s$. Therefore we can write

$$\frac{{}_nC_{n-s}}{n-s} = (-1)^{(n-1)(s+1)} \cdot \frac{{}_nC_s}{s}.$$

Now the exponent $(n - 1)(s + 1)$ is an even number except when n and s are both even. But, by Theorem II, when n and s are both even the circulants are both zero. Hence we can drop the sign factor and write the formula as in the statement of the theorem.

THEOREM IV. *When $s < \frac{1}{2}n$, ${}_nC_s = {}_{n-s}C_s$.*

If we add to the last $s - 1$ rows, respectively, the rows obtained by subtracting the first row from the next $s - 1$ rows, we shall have a determinant in which all the elements below the principal diagonal in the first s columns are zeros. Hence the first s columns and the first s rows may be dropped, and the circulant ${}_nC_s$ is thus reduced to the circulant ${}_{n-s}C_s$.

By repeating this process, if necessary, we can get

$$ {}_nC_s = {}_{n-s}C_s = {}_{n-2s}C_s = \cdots = {}_{s+r}C_s,$$

where r is the remainder when n is divided by s.

THEOREM V.[1] *When n and s are prime to each other, ${}_nC_s = s$.*

By Theorems IV and III we can reduce the subscripts n and s alternately until we have 1 for s, and a circulant equal to 1. When by Theorem III we reduce the second subscript, say from s to s', we introduce the factor s/s'. The next factor so introduced has s' in the numerator, and so each numerator cancels the preceding denominator until we come to the last factor, which has 1 for denominator. Thus we have finally only the factor s, and ${}_nC_s = s$.

THEOREM VI. *${}_n\Delta_s = 0$ or $s \cdot {}_n\Delta$, according as n and s have or do not have a common factor.*

This follows at once from Theorems I, II, and V.

COROLLARY. *If m rows of a determinant Δ be added cyclically, s in a set, the determinant thus obtained will be equal to 0, or $s\Delta$, according as m and s have or do not have a common factor.*

Other interesting results can be found in a similar manner by taking other circulants.

The formula $\Delta_s = s\Delta$ furnishes a simple solution for the following geometrical exercises:

Let K represent the area of the triangle whose vertices are the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and K' the area of the triangle whose vertices are the middle points of its sides.

If we take the determinant

$$ {}_3\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} $$

which is equal to $2K$, ${}_3\Delta_2$ with its elements all divided by 2 will be the determinant form of $2K'$. Therefore, since ${}_3\Delta_2 = 2 \cdot {}_3\Delta$, we have $K = 4K'$.

Similarly, if V represents the volume of a tetrahedron expressed in terms of the coördinates of its vertices, and V' the volume of the tetrahedron whose vertices are the centroids of its faces, we have $V = 27V'$, by using the formula for ${}_4\Delta_3$. Furthermore, the fact that ${}_4\Delta_2 = 0$ shows that the middle points of four consecutive edges, say P_1P_2, P_2P_3, P_3P_4, P_4P_1, lie in a plane. But if we add the first three rows of ${}_4\Delta$ in sets of two and then divide these rows by 2, we get $V = 4V_1$ (Corollary under Theorem VI), where V_1 represents the volume of the tetrahedron formed by joining one vertex to the middle points of the edges of the opposite face.

The determinant ${}_n\Delta_2$ is one in which each element is a binomial. Hence it can be expressed as the sum of 2^n determinants in which each element is a monomial. That one in which all the columns are made up from the first terms of these binomials is equal to Δ, and that one in which all the columns are from second terms is equal to $\pm\,\Delta$, according as n is odd or even. But ${}_n\Delta_2 = 2\Delta$, or 0, according as n is odd or even. Therefore the sum of all the other determinants must vanish. We wish now to determine in what manner they vanish.

[1] Muir says (*Theorie of Determinants*, vol. 2, London, 1911, p. 403.) this theorem was proved by Catalan in 1846, but Muir himself does not give the proof.

To study this phase of the subject we will multiply Δ by the general circulant whose first row is $k_1\ k_2\ k_3 \cdots k_n$, and in the product determinant we will write the terms in all the polynomial elements in the order of the subscripts of the k's. Then in the first row the subscripts of the letters which come from the respective columns of Δ will be in the same order, but the subscripts of these letters in the other rows will be those obtained from the first row by a cyclic permutation.

For example, when $n = 3$, we have

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ k_3 & k_1 & k_2 \\ k_2 & k_3 & k_1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} k_1a_1 + k_2a_2 + k_3a_3, & k_1b_1 + k_2b_2 + k_3b_3, & k_1c_1 + k_2c_2 + k_3c_3 \\ k_1a_2 + k_2a_3 + k_3a_1, & k_1b_2 + k_2b_3 + k_3b_1, & k_1c_2 + k_2c_3 + k_3c_1 \\ k_1a_3 + k_2a_1 + k_3a_2, & k_1b_3 + k_2b_1 + k_3b_2, & k_1c_3 + k_2c_1 + k_3c_2 \end{vmatrix}.$$

This product determinant can be expressed as the sum of n^n determinants with monomial elements. In all of these determinants the subscripts of the k's, and also those of the other letters in the first rows, form the n^n permutations that can be made with the first n consecutive integers, allowing all possible repetitions, and assuming that these other letters are always written in the same alphabetic order as in Δ. Since in each case all the elements in any one column are multiplied by the same k, these k's may be factored out leaving a determinant in which the columns are always in the order $a\ b\ c \cdots$, while the subscripts in any column are in direct cyclic order.[1]

Those determinants, the subscripts of whose first rows include all the permutations that can be made from a given combination of subscripts, are multiplied by a product of k's having the same combination of subscripts. Their sum may therefore be taken as the coefficient of this product of k's. To represent this sum let us write the subscripts of the first row of one of these determinants in a parenthesis, preceded by the number of permutations that can be made with the numbers within the parenthesis. The sum of the numbers within the parenthesis is the same as the sum of the numbers formed by multiplying together the subscript of each k and its exponent. This number I shall call the *weight* of the symbol, or of the determinants, and also the *weight* of the term in the k's. With this notation the above product determinant can be written

$$(111)k_1^3 + (222)k_2^3 + (333)k_3^3 + 3(112)k_1^2k_2 + 3(223)k_2^2k_3 + 3(331)k_3^2k_1 + 3(122)k_1k_2^2 + 3(233)k_2k_3^2 + 3(311)k_3k_1^2 + 6(123)k_1k_2k_3.$$

If, now, in any one of these symbols we change n to 1, and add 1 to each of the other numbers, we get another symbol, equal to the first when n is odd, and differing only in sign when n is even. For by this operation we merely permute cyclically the rows in each of the determinants represented by the symbol, that is, we carry the top row to the bottom. By repeating this process we can always

[1] In the circulant, however, the subscripts in any column are in reverse cyclic order.

get n equal symbols if n is odd and prime. If n is composite there may be fewer than n symbols in the group. For example, there are only two symbols in the group 90(113355), 90(224466). If n is even the symbols in any group will be alternately positive and negative. Hence the expansion can be collected into groups, the number of terms in each group being n or some factor of n, and each group can be written as a polynomial in the k's alone, multiplied by the sum of determinants represented by one of these symbols, that is, by one of these symbols itself.

When the above expansion, for example, is thus written it becomes

$$(111)(k_1^3 + k_2^3 + k_3^3) + 3(112)(k_1^2k_2 + k_2^2k_3 + k_3^2k_1) + 3(122)(k_1k_2^2 + k_2k_3^2 + k_3k_1^2) + 6(123)k_1k_2k_3.$$

If, however, we expand the circulant itself, collect its terms into groups, and multiply each group by Δ, we get another form of expansion for identically the same product, since the k's are arbitrary. Hence the coefficients of the corresponding groups in these two expansions must be equal, and thus we get the value of any sum of determinants represented by one of our symbols. Such a sum must, therefore, be equal to a multiple of Δ, or to zero, according as the corresponding group of terms is present in, or absent from, the expansion of the circulant.

In the case of the above example, $n = 3$, this second method of expansion gives $(k_1^3 + k_2^3 + k_3^3)\Delta - 3k_1k_2k_3\Delta$. Therefore we have $(111) = \Delta$, [(111) is of course Δ as given.]

$$3(112) = 3(122) = 0, \quad \text{and} \quad 6(123) = -3\Delta.$$

THEOREM VII. *The sum of the determinants represented by one of these symbols is equal to zero when the weight is not a multiple of n.*

To prove this it will be sufficient to show that the weight of every term in the expansion of a circulant of order n is a multiple of n.[1]

If any minor, whose principal diagonal coincides with that of the circulant, be expanded in terms of the elements of its first column and their cofactors, it can readily be seen that the weights of its terms are either 1 or $n + 1$ greater than the weights of the terms in such a minor of the next lower order. Then, beginning with k_1 in the lower right-hand corner, and ending with the circulant itself, we find that the weight of every term is indeed a multiple of n.

COROLLARY. *When n is even the sum of the determinants denoted by* $n!(123\cdots n)$ *is equal to zero.*

THEOREM VIII. *When the weight of one of these symbols is a multiple of n, the sum of the determinants it represents is of the form* $\pm\lambda n\Delta$, *where* λ *is some integer, including zero, when n is prime; when n is composite* λ *is either an integer, including zero, a fraction whose denominator is a divisor of n, or the sum of two or more such numbers. In the special cases*

$$(11\cdots1) = \pm(22\cdots2) = (33\cdots3) = \cdots = \pm(nn\cdots n) = \Delta, \lambda = 1/n.$$

[1] This was proved by R. Baltzer about 1870. (Muir, vol. 3, p. 376.)

To prove this it will be sufficient to show that the coefficients in the expansion of the circulant are numbers like λn as defined above.

The value of these coefficients is determined by the fact that the elements in the principal diagonal, and also in any diagonal parallel to it, are all alike. Hence if we start from each element in any term and take the next element in the same cyclic order in the same diagonal, we shall obtain the same term. If n is prime, we can repeat this operation n times before reaching the term from which we started. That is, there will be n terms in such a cycle. But if $n = pq$, say, such a cycle may consist of n terms, or of only p terms, or of only q terms. Moreover, all the terms in any cycle have the same sign, for at each step the column and row numbers of each element are both increased by 1, except that in passing from the last row to the first row, and from the last column to the first column, two of these numbers are changed from n to 1. The first of these operations makes no change in the number of inversions. The second makes an even number of such changes. Therefore the sign is not changed. The coefficients of the terms $k_1{}^n \cdots k_n{}^n$, given by the principal diagonal and those parallel to it, are evidently either 1, or -1.

The same terms in different cycles may not have the same sign, and so may wholly or in part cancel one another.

COROLLARY I. *When n is odd the sum of the determinants denoted by $n!(123\cdots n)$ is equal to $\lambda n\Delta$, where λ is an integer.* For when there is no letter repeated in a term of the circulant there are always n terms in each cycle.

When n is 3, 5, 7, 9, I have found these values to be, respectively,

$$(0-1)3\Delta = -3\Delta, \quad (1-2)5\Delta = -5\Delta, \quad (2-17)7\Delta = -105\Delta, \quad (113-98)9\Delta = 135\Delta,$$

where the first number in the parenthesis is the number of positive cycles, and the second is the number of negative cycles, in which the term $k_1k_2k_3 \cdots k_n$ occurs in the expansion of the circulant.

COROLLARY II. *If all possible cyclic permutations be made of the elements in each column of a given determinant, and the n^n determinants thus formed be collected into groups such that each determinant in the group shall have the same weight, and the same subscripts in the first row, these groups will be the same as those represented by the above symbols. Hence the sum of any group will be given by Theorem VIII, or Theorem VII, according as its weight is, or is not, divisible by n.*

Groups that can be obtained from each other by making the same cyclic permutations of whole rows in each determinant of the group are equal, or differ only in sign, according as n is odd or even.

THEOREM IX. *When n is odd the sum of the $(n-1)!$ determinants of the sum denoted by $n!(123\cdots n)$ which have one column in common is equal to $\lambda\Delta$, where λ is an integer.*

Take the $(n-1)!$ determinants in which the first columns are the same as the first column of Δ. Permute cyclically the rows in each determinant of this group and we have a second group of determinants all having the same first column and

each equal to the corresponding determinant of the first group, n being odd. Repeating this operation we get n equal groups of this kind, all together comprising the $n!$ determinants of the sum $n!(123\cdots n)$. As the entire sum is equal to $\lambda n\Delta$, the sum of the determinants in any one group is equal to $\lambda\Delta$.

We could start with any other column and proceed in the same way.

Thus when $n = 3$ the six determinants of 6(123) can be written

$$\begin{vmatrix} a_1 & b_2 & c_3 \\ a_2 & b_3 & c_1 \\ a_3 & b_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_2 & b_3 & c_1 \\ a_3 & b_1 & c_2 \\ a_1 & b_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_3 & b_1 & c_2 \\ a_1 & b_2 & c_3 \\ a_2 & b_3 & c_1 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_3 & c_2 \\ a_2 & b_1 & c_3 \\ a_3 & b_2 & c_1 \end{vmatrix} = \begin{vmatrix} a_2 & b_1 & c_3 \\ a_3 & b_2 & c_1 \\ a_1 & b_3 & c_2 \end{vmatrix} = \begin{vmatrix} a_3 & b_2 & c_1 \\ a_1 & b_3 & c_2 \\ a_2 & b_1 & c_3 \end{vmatrix}.$$

Since the total sum of the six determinants is -3Δ, the sum of the determinants in each column must be $-\Delta$.

When n is even we can form n groups in the same way and the sum of the determinants in one group will be numerically equal to the sum in any other group, but the groups in which this sum is positive will cancel the groups in which this sum is negative and the total sum will be zero. Therefore we cannot find the sum of those in any one group in this way.

When $n = 4$, for example, I have found the sum of the six determinants, in each of which the first column is the first column of Δ, to be

$$-2\begin{vmatrix} a_3 & b_1 & c_1 & d_1 \\ a_4 & b_2 & c_2 & d_2 \\ a_1 & b_3 & c_3 & d_3 \\ a_2 & b_4 & c_4 & d_4 \end{vmatrix},$$

which is not a multiple of Δ.

SOME PROPERTIES OF A SKEWSQUARE.[1]

By W. H. ECHOLS, University of Virginia.

A plane quadrilateral in which two diagonals are perpendicular and equal to each other has a number of rather interesting properties. Such a figure for convenience of reference in the present paper will be called a *skewsquare*, although I am given to understand it has at some time been called a pseudo-square.[2] The writer has seen no reference to it in print and it is possible that some or all of the properties enunciated below are not new. A number of the properties admit of easy demonstration by elementary geometry and also as easy exercises in interpreting complex number relations. Furthermore the figures involved in the geometrical constructions furnish excellent exercises in mechanical drawing inasmuch as the constructions check themselves as to accuracy in various ways.

[1] Presented at the Cambridge meeting of the Association, December, 1922.

[2] For example *Mathesis*, 1894, p. 268.—EDITORS.

The writer was led to the consideration of this type of quadrilateral in trying to place the complex number

$$w_{ij} \equiv \frac{1}{z_j - z_i} \int_{z_i}^{z_j} f(z)dz,$$

with reference to $f(z)$ in such a manner as to locate ζ that $f(\zeta) = w_{ij}$, wherein $f(z)$ is an analytic function. To any pair of points in the z-plane corresponds a point in the w-plane, the mean value of $f(z)$.

To the quadrilateral points z_1, z_2, z_3, z_4 correspond the quadrilateral points w_{12}, w_{23}, w_{34}, w_{41}, and in this figure it can be easily shown (by the properties of the integral) that the triangles $w_{12}w_{23}w_{31}$ and $w_{13}w_{34}w_{41}$ are, respectively, similar to $z_1z_2z_3$ and $z_3z_4z_1$. Also triangles $w_{24}w_{32}w_{34}$ and $w_{12}w_{24}w_{41}$ are, respectively, similar to $z_2z_3z_4$ and $z_1z_2z_4$. In particular, if the four z-points are the corners of a square, then the corresponding four w-points are the corners of a skewsquare. In like manner, with n z-points taken around a polygon can be constructed a skew-polygon of n w-points, and in addition the w-points corresponding to the diagonals of the z-polygon give a figure with more complicated but interesting relations.

The following are some properties of skewsquares:

(1) In a skewsquare the two squares constructed on two opposite sides as diagonals have a common vertex. The two points thus determined will be called the *foci* of the skewsquare.

(2) Conversely, if the squares constructed on two opposite sides of a quadrilateral as diagonals have a common vertex, then will the squares on the other two sides as diagonals also have a common vertex, and the figure is a skewsquare. Otherwise, if any two isosceles right triangles have the vertex common, their bases are opposite sides of a skewsquare.

(3) The foci and the mid-points of the diagonals of a skewsquare are the corners of a square (called the focal square). The center of this square will be called the *center* of the skewsquare. The mid-points of the diagonals of a skewsquare will be called the *conjugate foci* of the skewsquare.

(4) The foci lie on the bisectors of the angles between the diagonals of the skewsquare, respectively.

(5) The center of a skewsquare is the centroid of its corners, and also the centroid of the corners of the four squares having for diagonals the sides of the skewsquare.

(6) The segment joining a focus to the midpoint of a side of a skewsquare is perpendicular to and equal to half a side.

(7) The sum of the squares of the opposite sides of a skewsquare are equal.

(8) The four triangles (not right angled) whose bases are the sides of a skewsquare and whose vertices are the foci are equal in area.

(9) Segments joining any vertex of a skewsquare to the two foci make equal angles with the sides at that vertex.

(10) The foci of a skewsquare are the foci of a conic tangent to the sides of the

skewsquare. The points of contact are points in which the bisectors of the angles between the diagonals cut the sides. A point of contact divides the side in the ratio of the adjacent segments of the diagonals determined by their intersection.

(11) The midpoints of the sides of a skewsquare form a square (called the midsquare), its side is equal to half the diagonal of the skewsquare. The circle circumscribing the midsquare (called the midcircle) is the auxiliary circle on the transverse axis of the tangent conic. The square of the transverse diameter of the conic is equal to half the square of the diagonal of the skewsquare.

(12) The area of each of the four triangles in (8) is equal to the square of the semi-conjugate diameter of the tangent conic. The foci are both inside, on, or outside the boundary of the skewsquare, the tangent conic is respectively an ellipse, a line-segment, or a hyperbola. Also the product of two opposite sides of a skewsquare and the cosine of the angle between them is equal to the square of the conjugate diameter of the conic.

(13) The sides of all skewsquares, whose diagonals are perpendicular diameters of two fixed equal circles, are tangent to a fixed conic. The power of the center with respect to either circle is equal to the sum of the squares of the semiaxes of the conic.

Otherwise, the locus of the vertices of skewsquares circumscribing a given central conic is two equal circles whose centers are the other corners of a square on the segment joining the foci as diagonal, the radius of the circles is equal to the side of a square inscribed in the auxiliary circle of the conic.

(14) When one side of a skewsquare is fixed and the diagonals become infinite in given directions the conic becomes a parabola tangent to two right-angled sides at the ends of the latus rectum.

(15) The four corners of the four squares having as diagonals the sides of a skewsquare (and which are not foci) are the corners of a second skewsquare, called the conjugate skewsquare (the first being called the primitive skewsquare). The square of the diagonal of the conjugate is twice that of the primitive skewsquare. The vertices of the conjugate lie on the bisectors of the angles between the diagonals of the primitive, the foci and the conjugate foci of the primitive are respectively the conjugate foci and the foci of the conjugate skewsquare.

Otherwise, the construction of the primitive from the conjugate skewsquare follows. The vertices of four squares whose diagonals are the four segments joining each vertex of a skewsquare to the midpoint of the diagonal not containing that vertex determine only four points which are the corners of another skewsquare (called the primitive skewsquare). Its vertices lie on the bisectors of the angles between the diagonals of the conjugate, the foci and the conjugate foci of the conjugate skewsquare are respectively the conjugate foci and the foci of the primitive.

Thus, associated with any given skewsquare there is a conjugate and a primitive skewsquare which are confocal, their diagonals are collinear and their midpoints coincide, the diagonal of the first being twice that of the second.

Associated with any skewsquare there is a series of derived skewsquares, alternate members are confocal and their tangent conics are members of a confocal system.

(16) Segments joining the center to the vertices of a primitive skewsquare are perpendicular to and equal to half the corresponding sides of its conjugate skewsquare.

(17) The diagonals of a skewsquare are parallel and equal to the diagonals of the midsquare of the conjugate skewsquare.

(18) Lines through the midpoints of the sides of a skewsquare parallel to the transverse axis of the tangent conic pass through the corners of its primitive skewsquare. The segments between these midpoints and the corners of the primitive skewsquare are equal to each other and to half the distance between the foci of this conic.

(19) The center of a skewsquare is the radical center of the four circles whose centers are the corners of the skewsquare, the radius of each circle is equal to the segment joining its center to either of the two adjacent corners of the primitive skewsquare.

(20) If Z is any point in the plane of a skewsquare $ABCD$ and ZA, ZB, ZC, ZD be rotated about Z, in the same direction, through one, two, three, four right angles, respectively, to ZA', ZB', ZC', ZD', then will the last four segments be in equilibrium.

(21) The four points which divide the four sides of a skewsquare in the same given ratio $(m : n)$ are the corners of a skewsquare (m and n may be real or complex numbers).

(22) If $ABCD$ and $A'B'C'D'$ are any two skewsquares, then the four points dividing the segments AA', BB', CC', DD' in given ratio ($m : n$ real or complex) are the corners of a skewsquare.

(23) If the corners of any one of a system of confocal skewsquares are the roots of a polynomial, the roots of its second derivative and the foci are the corners of a lozenge composed of two equilateral triangles.

(24) If s_1, s_2, s_3, s_4 are the lengths of the segments into which the intersection of diagonals of a skewsquare divides those diagonals, then the two foci and the intersection of diagonals of the skewsquare (whose corners are z_1, z_2, z_3, z_4) are the roots of the derivative of the function

$$(z - z_1)^{s_1}(z - z_2)^{s_2}(z - z_3)^{s_3}(z - z_4)^{s_4}.$$

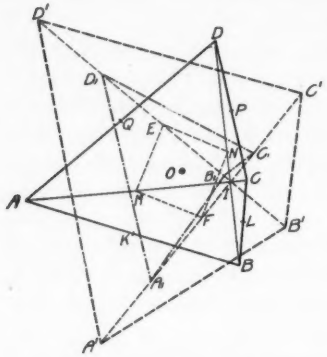
(25) If r_1, r_2, r_3, r_4 are the segments joining any point Z to the corners of a skewsquare, and s_1, s_2, s_3, s_4 are defined as in (24), then forces acting at Z whose directions pass through the corners and whose magnitudes are proportional to

$$\frac{s_1}{r_1} : \frac{s_2}{r_2} : \frac{s_3}{r_3} : \frac{s_4}{r_4}$$

are in equilibrium when Z is at either focus of the skewsquare.

The following abbreviated proofs are offered to establish the properties

enunciated above. In the accompanying diagram, $ABCD$ is a skewsquare, the diagonals intersect at I, the foci are E and F, M and N are the midpoints of diagonals and K, L, P, Q the midpoints of sides. The figure is drawn for a convex skewsquare, it may however be either reëntrant or two opposite sides may cut internally. $A'B'C'D'$ is the conjugate, and $A_1B_1C_1D_1$ the primitive of $ABCD$.



(1) The diagonal AC can be brought to BD by a rotation through a right angle, first A to B and C to D by rotation about E, second A to D and C to B by rotation about F.

(2) In the two right isosceles triangles AEB, CED the rotation AEC about E through a right angle brings A to B and C to D and AC to BD at right angles. There is a second center of rotation F accomplishing the same result, as in (1).

(3) The triangles AEC and BED are congruent, EM turns through a right angle to EN, in like manner FM turns through a right angle to FN. Hence $EMFN$ is a square.

(4) The altitudes of the congruent triangles AEC, BED are equal, hence E (and in like manner F) is equidistant from the diagonals.

(5) The mid-points of the sides of a skewsquare are the corners of a square called the mid-square. Its sides are obviously parallel and equal to half the diagonals of the skewsquare. The centroid of the corners of the skewsquare is obviously the center of the mid-square, and the centroid of the foci is O the mid-point of MN which is also the centroid of the corners of the skewsquare. The centroid of the corners of the four squares on the sides of the skewsquare as diagonals is the centroid of the corners of the mid-square.

(6) This is obvious when the focus is the vertex of the right isosceles triangle having a side as the base. In any other case such as EQ, since QL and EF bisect each other at O, then QE is parallel and equal to FL which is perpendicular and equal to half BC.

(7) $AB^2 + CD^2 = AI^2 + BI^2 + CI^2 + DI^2 = CB^2 + AD^2$.

(8) The triangles AED and BEC have equal area, for AE is equal and perpendicular to EB, ED is equal and perpendicular to EC. The included angles are therefore equal or supplementary.

(9) $\angle DAB$ is half a right angle plus either $\angle EAD$ or $\angle FAB$ (if E and F are outside use *minus*).

(10) The property in (9) is a well-known fundamental property of a conic having foci E, F and tangent to the sides of the skewsquare. The circumcircle of the square $AEBA'$ passes through I, therefore IA' makes with IA half a right angle and contains F, by (4). FA' cuts AB in T such that

$$\angle KTA' = \angle ETK = \angle FTB.$$

Since the focal radii to the point of contact make equal angles with the tangent, T is the point of contact. Since IT bisects AIB, T divides AB in ratio $AI : BI$.

(11) The sum (difference) of the focal radii to the point of contact is equal to FA'. Also PF is parallel and equal to $EK = KA'$, (6), each being perpendicular and equal to $\frac{1}{2}AB$. Therefore FA' is parallel and equal to KP. Hence the midcircle is the auxiliary circle of the tangent conic. Also $KP^2 = 2KQ^2 = \frac{1}{2}DB^2$.

(12) The product of the perpendicular from F on AB and $EK = KB$ is equal to the square on the semi-minor axis, and to the area of AFB which is equal to

$$\tfrac{1}{2}AF \cdot FB \sin AFB = \tfrac{1}{4}AD \cdot BC \sin AFB.$$

$\angle AFB + \angle LFQ = 2\pi - \frac{1}{2}\pi = \frac{3}{2}\pi$, $\angle LFQ = \pi - \angle (AD, BC)$. Therefore $\angle AFB = \angle (AD, BC) + \frac{1}{2}\pi$. Therefore $\sin AFB = \cos \angle (AD, BC)$. This establishes the statement. Obviously the eccentricity of the conic is equal to the ratio of the radius of the focal circle to that of the midcircle. In virtue of the triangles in (8) the foci E and F must be inside, on, or outside the boundary of the skewsquare, the areas of the triangles being counted positive for inside foci and negative for outside foci.

(13) The midpoints M and N are fixed and therefore the foci are also. The diagonal being of constant length, the transverse axis is fixed in length and position, and therefore the conic which the sides touch is fixed. The power of O with respect to the circle on AC as diameter is equal to

$$AM^2 - OM^2 = QP^2 - OM^2 = 2OP^2 - OM^2 = OP^2 + (OP^2 - OE^2).$$

We note by holding the midsquare and the focal circle fixed we derive the forms of skewsquares touching congruent concentric conics. If only the midsquare be fixed the skewsquares touch concentric conics having equal transverse axes. When the foci are on the boundary the skewsquare degenerates into an isosceles right triangle, one focus is the midpoint of the hypothenuse and the other is at the vertex of the right angle and the conic degenerates into the segment joining them. When two opposite sides of the skewsquare cut internally the conic is an hyperbola, since the foci lie on opposite sides of a tangent.

(14) If AB is fixed and the diagonals extend indefinitely intersecting in I, the other two sides whose ends are A and B become parallel to the diagonals and cut at right angles, in say Z. The single focus is E and the bisector IE makes equal angles, $\frac{1}{4}\pi$, with the right-angled sides, and ZE makes this same angle with these sides. This establishes the statement from the known properties of the parabola.

(15) In the figure, the circle on AB as diameter passes through A' and I, therefore AIA' is half the right angle AKA' and A' is on bisector IF, and so for each corner. By (16) FP is parallel and equal to EK, hence KF is parallel and equal to EP or $\frac{1}{2}EC'$. Therefore from similar triangles $A'KF$ and $A'EC'$, we have $A'F = FC'$. Also $A'F$ is parallel and equal to KP.

$$A'C'^2 = 4KP^2 = 8KQ^2 = 2BD^2.$$

In like manner $B'D'^2 = 2AC^2$, and E is the midpoint of $B'D'$. Hence M, N are the foci of $A'B'C'D'$.

(16) A_1O is parallel and equal to $\frac{1}{2}A'E$, therefore A_1O is perpendicular and equal to $\frac{1}{2}AB$.

(17) A_1C_1 is parallel and equal to $\frac{1}{2}KP$, by (18).

(18) $A'F = BN\sqrt{2}$ and $BN = A_1F\sqrt{2}$. Therefore $A'F = 2A_1F$. $A'A_1 = = A_1F$, $A'K = KE$. Therefore A_1K is parallel and equal to $\frac{1}{2}FE$, and so for the other points.

(19) $AA_1 = AD_1$, $BA_1 = BB_1$. The circles, centers A and B having radii AA_1, BB_1, respectively, have their radical axis passing through $A_1 \perp AB$ and therefore passing through O, by (16).

(20) The characteristic property defining a quadrilateral $ABCD$ whose corners are the complex numbers z_1, z_2, z_3, z_4, as a skewsquare is

$$z_3 - z_1 = i(z_2 - z_4),$$

which can be written

$$z_1 + \omega z_2 + \omega^2 z_3 + \omega^3 z_4, \tag{1}$$

where ω is the principal fourth root of unity, i. Any four numbers which satisfy this equation are the corners of a skewsquare, this then is the equation of a skewsquare and it establishes at once the truth of (20) on multiplying (1) through by i.

(21) If the corners of a skewsquare satisfy equation (1) in (20), then the same points satisfy the equation

$$z_2 + \omega z_3 + \omega^2 z_4 + \omega^3 z_1 = 0. \tag{2}$$

On multiplying (1) by $m/(m + n)$ and (2) by $n/(m + n)$ and adding, the resulting equation proves the statement in (21). If m, n are real numbers, division of a segment in ratio $m : n$ is conventional and familiar. When m, n are complex numbers then the point w "dividing the segment" z_1 to z_2 in ratio $m : n$ is determined, as before, from $w = (mz_1 + nz_2)/(m + n)$, or what is the same thing

$$\frac{w - z_2}{z_1 - w} = \frac{m}{n}.$$

The point w is therefore the vertex of a triangle constructed on segment z_1 to z_2 as base similar to the triangle whose sides are m and n. Therefore, if similar triangles be similarly constructed (all outwards or all inwards) on the sides of a skewsquare as bases, then the vertices are the corners of a skewsquare.

(22) Let equation (1) in (20) be any skewsquare and $z_1' + \omega z_2' + \omega^2 z_3' + \omega^3 z_4' = 0$ be any other skewsquare. Then on multiplying the first equation by $m/(m + n)$ and the second by $n/(m + n)$ and adding, the statement is verified.

(23) Take the origin at the center. Let ζ and η represent the midpoint of and half of a diagonal respectively. Then the corners are the roots of

$$(z - \zeta - \eta)(z - \zeta + \eta)(z + \zeta - i\eta)(z + \zeta - i\eta) = 0,$$

or

$$(z^2 - \zeta^2)^2 - 4\zeta\eta^2 z - \eta^4 = 0.$$

The roots of the second derivative are $z = \pm \zeta/\sqrt{3}$.

(24) Let $f(z)$ be the function. Take the logarithm and differentiate. Then

$$f'(z) = f(z)\left(\frac{s_1}{z - z_1} + \frac{s_2}{z - z_2} + \frac{s_3}{z - z_3} + \frac{s_4}{z - z_4}\right).$$

The roots of $f'(z)$, not common to $f(z)$, are gotten by the vanishing of the parenthesis. Also

$$s_1 + s_3 = s_2 + s_4 = d,$$

the length of the diagonal. Also

$$\frac{s_1 z_3 + s_3 z_1}{s_1 + s_3} = \frac{s_2 z_4 + s_4 z_2}{s_2 + s_4} = I,$$

the intersection of the diagonals. Clearing the equation of fractions and dividing by d, there results $(z - I)[(z - z_1)(z - z_3) + (z - z_2)(z - z_4)] = 0$. The quadratic in the square bracket is $z^2 - \frac{1}{2}(z_1 + z_2 + z_3 + z_4)z + \frac{1}{2}(z_1 z_3 + z_2 z_4) = 0$. Transfer the origin to the centroid (center). Then $z_1 = \zeta + \eta$, $z_3 = \zeta - \eta$, $z_2 = -\zeta - i\eta$, $z_4 = -\zeta + i\eta$, $\zeta \equiv OM$, $\eta = MA$. Then $z_1 z_3 + z_2 z_4 = 2\eta^2$. The equation then becomes $(z - I)(z + i\zeta)(z - i\zeta) = 0$, as required.

(25) In the parenthesis in (24) put $z - z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$, etc. Equating to zero the real and imaginary components shows that the sum of the components along two right-angled axes vanishes, and so also must their resultant, and the segments are in equilibrium, when Z is at either focus.

THE MINIMAL PROPERTIES OF THE ISOGONAL CENTERS OF A TRIANGLE.

By H. M. LUFKIN, Dunkirk, N. Y.

An isogonal center[1] of a triangle may be defined as a point such that if lines are drawn *through* this point to the vertices of the triangle, the angular space around this point is divided into six equal angles of 60° each. There are two types of isogonal centers, one of which we shall call an I_+ point and the other an I_- point; the distinction being as follows: Let P_1, P_2, and P_3 be the vertices of the triangle taken in counter-clockwise order and draw the lines l_1, l_2, and l_3 through the isogonal center to the respective vertices; then if, as we go around the isogonal center in counter-clockwise order, we meet these lines in the order l_1, l_3, l_2, the center is an I_+ point; if in the order l_1, l_2, l_3, it is an I_- point.

The isogonal centers of a triangle may be constructed by drawing equilateral triangles on the sides of the given triangle and drawing the circumcircles of these

[1] Neuberg, *Mémoires Couronnés, L'Académie Royale de Belgique*, vol. 44, 1891, p. 12ff. M'Cay, *Mathesis*, vol. 7, 1887, p. 211ff.

equilateral triangles. These circumcircles will intersect in a point which is an isogonal center of the triangle. If the equilateral triangles are drawn outwardly, the point will be an I_+ point; if inwardly, an I_- point. From this method of construction the following facts may readily be seen:

Every triangle has a unique I_+ and a unique I_- point; except the equilateral triangle which has a unique I_+ point, but whose I_- point is anywhere on the circumcircle of the triangle.

Where one angle of the triangle is 120°, it is convenient to define the I_+ point as the vertex of this angle; and where *only one* angle of the triangle is 60°, it is convenient to define the I_- point as the vertex of this angle.

Where no angle of the triangle is greater than or equal to 120°, the I_+ point is inside the triangle, and the *half* lines from this point to the vertices divide the angular space about the point into angles of 120°, 120°, and 120°; where one angle of the triangle is greater than 120°, the I_+ point is outside the triangle and the half lines from the point to the vertices of the triangle divide the angular space about the point into angles of 60°, 60°, and 240°.

The I_- point is always outside the triangle, except in the 60° case noted above, and the half lines from this point to the vertices make angles of 60°, 60° and 240°.

A problem which has been discussed by various writers[1] is that of determining a point P_0 such that the sum of the distances ρ_1, ρ_2, ρ_3 from this point to the vertices of a triangle P_1, P_2, P_3, respectively, shall be a minimum. It has been shown that if no angle of the triangle is greater than 120°, the point P_0 is the I_+ point; and when one angle is greater than 120°, the point P_0 is at the vertex of the obtuse angle.

In this paper a further investigation of this problem will be made to determine whether the I_+ and I_- points possess any further minimum properties when we consider the functions $F_{++-} = \rho_1 + \rho_2 - \rho_3$, $F_{+-+} = \rho_1 - \rho_2 + \rho_3$, and $F_{-++} = -\rho_1 + \rho_2 + \rho_3$.[2] The functions where all the signs are negative or where two are negative need not be considered since it is evident that they have no absolute minimum.

Since the I_+ and I_- points have peculiar properties for triangles having angles of 60° and 120°, we shall divide all triangles into the following classes:

Class	I	II	II′	III	IV	V
Number of angles						
$< 60°$	1	1		2	2	2
$= 60°$		1	3			
$> 60°$ and $< 120°$	2	1		1		
$= 120°$					1	
$> 120°$						1

[1] Cavalieri, *Exercitationes Geometrica*, 1647, pp. 504–519. D. Jackson, this MONTHLY, *1917*, 42–44. R. A. Johnson, this MONTHLY, *1917*, 243–244. Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, pp. 130–131. De la Vallée Poussin, *Cours d'Analyse*, vol. 1, third edition, Louvain, 1914, pp. 165–166.

[2] The cases of several of these functions are mentioned by Jacob Steiner, *Gesammelte Werke*, vol. 2, p. 729, but his results are erroneous.

The functions F_{++-}, F_{+-+}, and F_{-++} are continuous all over the plane, and $\partial F/\partial x$ and $\partial F/\partial y$ exist everywhere except at the vertices of the triangle. If we enclose a portion of the plane around the triangle, then there is for each of these functions at least one point within this region or on its boundary where this function takes on its minimum value for the region. It is evident that if we enclose the triangle with a large boundary, say a circle with its center at the circumcenter of the triangle and a radius a hundred times the radius of the circumcircle, that as we move in from the boundary the value of the function is less than on or outside the boundary; so that the minimum does not occur on the boundary, and must therefore occur at some point within the region. Since this is the case, the minimum occurs at a point where the first derivatives are zero, or where they do not exist.

Let the vertices of the triangle be $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$. The function F_{-++} is then

$$F(x, y) = -\sqrt{(x-x_1)^2+(y-y_1)^2} + \sqrt{(x-x_2)^2+(y-y_2)^2} + \sqrt{(x-x_3)^2+(y-y_3)^2}$$

and denoting the three distances by ρ_1, ρ_2, ρ_3, the first partial derivatives are

$$\frac{\partial F}{\partial x} = -\frac{x-x_1}{\rho_1} + \frac{x-x_2}{\rho_2} + \frac{x-x_3}{\rho_3}, \qquad \frac{\partial F}{\partial y} = -\frac{y-y_1}{\rho_1} + \frac{y-y_2}{\rho_2} + \frac{y-y_3}{\rho_3}.$$

From the point (x, y) we shall draw a half-line in the direction of the positive end of the X-axis, and denote the angles which the half-lines to the three vertices make with this half-line, by φ_1, φ_2, and φ_3, respectively. We may then write $\partial F/\partial x = -\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3$, $\partial F/\partial y = -\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3$. From the equations

$$-\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3 = 0 \quad \text{and} \quad -\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3 = 0,$$[1]

we obtain

$$\varphi_3 - \varphi_2 = \pm 120°, \ \pm 240°, \tag{1}$$
$$\varphi_2 - \varphi_1 = \pm 60°, \ \pm 300°, \tag{2}$$
$$\varphi_1 - \varphi_3 = \pm 60°, \ \pm 300°. \tag{3}$$

There are sixteen possible combinations of these values obtainable from (1) and (2), but of these, only the following six combinations satisfy the conditions of (3):

	I	II	III	IV	V	VI
$\varphi_3 - \varphi_2 =$	240°	240°	120°	−240°	−240°	−120°
$\varphi_2 - \varphi_1 =$	−300°	60°	−60°	300°	− 60°	60°

For each of these six combinations it is seen that the half-line from (x, y) to the vertex P_1 bisects an angle of 120° made by the half-lines to the other two vertices.

[1] Cf. Steiner, *op. cit.*

We may express this by saying that the point (x, y) must satisfy condition C_1. A similar argument holds for the function F_{+-+}, and the first derivatives can exist and vanish only at a point which would fulfill condition C_2, namely, that the half-line from the point (x, y) to the vertex P_2 must bisect an angle of 120° made by the half-lines to the other two vertices. A similar condition C_3 must be fulfilled for the vanishing of the derivatives of the function F_{++-}.[1]

We have then shown that for each of the functions F_{++-}, F_{+-+}, and F_{-++} the absolute minimum exists and that it occurs at an I_- point or at a vertex of the triangle, except for triangles of Class V where it occurs at an I_+ point, an I_- point, or at a vertex of the triangle.

It will be seen from the table classifying triangles that in all classes except II and II′ there is one odd angle differing from the other two. From an examination of the various classes of triangles it is seen that the half-line from the I_- point which bisects the angle formed by the other two half-lines is the one to the vertex of this odd angle. If we specify that θ_3 shall be this odd angle; then for triangles of Classes I, III, IV, and V, there is no point which fulfills conditions C_1 or C_2, and hence for the functions F_{-++} and F_{+-+} the only points at which an absolute minimum can occur are the vertices of the triangle. For triangles of Classes I, III, and IV, condition C_3 is fulfilled at just one point, namely, the I_- point; and therefore the absolute minimum for the function F_{++-} can occur at the I_- point or at one of the vertices of the triangle. For triangles of Class V, condition C_3 is fulfilled at two points, namely, the I_+ point and the I_- point; and therefore the absolute minimum for the function F_{++-} can occur at either of the isogonal centers or at one of the vertices of the triangle. For triangles of Class II, since, strictly speaking, none of the conditions C_1, C_2, or C_3 are fulfilled anywhere in the plane, the absolute minimum for all of the functions must occur at one of the vertices of the triangle. For triangles of Class II′, any point on the arc of the circumcircle from P_2 to P_3 fulfills condition C_1; and therefore the absolute minimum for the function F_{-++} can occur at points of this arc or at the vertex P_1. Similarly, the absolute minimum for the function F_{+-+} occurs at points of the arc of the circumcircle from P_1 to P_3 or at the vertex P_2, and for the function F_{++-} at points of the arc from P_1 to P_2 or at the vertex P_3.

In cases where the absolute minimum must occur at one of the vertices, this vertex can be determined from an examination of the relative lengths of the sides of the triangle. In cases where the absolute minimum can occur at an I point or one of the vertices, it remains only to investigate the relative values of the function F_{++-} at the I_- point and at the vertices of the triangle for triangles of Classes I, III, and IV; and for triangles of Class V at the I_+ point, the I_- point, and the vertices. These results are shown in the appended tables.

Since the first derivatives of the function F_{++-} vanish at the I_- point for all

[1] Steiner (*op. cit.*) has shown that the condition for the vanishing of the first derivatives of the function F_{+++} is that the half-lines from the point (x, y) to the vertices of the triangle make three angles of 120° with each other.

triangles except those of Class II, and at the I_+ point for triangles of Class V, the question arises as to whether these points yield a relative maximum or minimum where they do not give the absolute minimum. An examination of the discriminant of the second derivatives shows, however, that these points are saddle-points.

For triangles of Class II, the first derivatives do not exist at the I_- point, which is the vertex of the 60° angle. In this case, the surface represented by the equation $z = F_{++-}(x, y)$ has a conical point corresponding to the I_- point. If we pass a plane through this conical point parallel to the Z-plane, the surface lies entirely above this plane, so that the I_- point is an absolute minimum as we have previously stated; but as we go out from this point in one and only one direction the surface starts tangent to the plane and then rises above it. In the case of the surface represented by the equation $z = F_{-++}(x, y)$ there is a conical point corresponding to the I_- point, and if we pass a plane through this conical point parallel to the Z-plane, the surface rises above this plane as we move away from the conical point in every direction except one; but as we move in this direction the surface starts tangent to the plane and then drops below it, giving a peculiar sort of saddle-point.

SUMMARY.

Character of the Isogonal Centers.

Angles of triangle.	I_+.	I_-.
$\theta_3 < 60° < \theta_2 \leqq \theta_1 < 120°$.....	Minimum F_{+++}	Minimum F_{++-}
$\theta_3 < 60° = \theta_2 < \theta_1 < 120°$.....	Minimum F_{+++}	Minimum F_{++-}
$\theta_1 = \theta_2 = \theta_3 = 60°$.....	Minimum F_{+++}	*See next table.*
$\theta_1 \leqq \theta_2 < 60° < \theta_3 < 120°$.....	Minimum F_{+++}	Saddle-point F_{++-}
$\theta_1 \leqq \theta_2 < 60°, \theta_3 = 120°$.....	Minimum F_{+++}	Saddle-point F_{++-}
$\theta_1 \leqq \theta_2 < 60°, \theta_3 > 120°$.....	Saddle-point F_{++-}	Saddle-point F_{++-}

Positions of the Absolute Minimum for the Various Functions and Classes of Triangles.

Angles of triangle.	F_{+++}.	F_{++-}.	F_{+-+}.	F_{-++}.
$\theta_3 < 60° < \theta_2 \leqq \theta_1 < 120°$	I_+	I_-	P_3	P_3
$\theta_3 < 60° = \theta_2 < \theta_1 < 120°$	I_+	$I_- = (P_2)$	P_3	P_3
$\theta_1 = \theta_2 = \theta_3 = 60°$.....	I_+	Arc of circumcircle from P_1 to P_2	Arc of circumcircle from P_1 to P_3	Arc of circumcircle from P_2 to P_3
$\theta_1 \leqq \theta_2 < 60° < \theta_3 < 120°$	I_+	P_1	P_1	P_2
$\theta_1 \leqq \theta_2 < 60°, \theta_3 = 120°$.	I_+	P_1	P_1	P_2
$\theta_1 \leqq \theta_2 < 60°, \theta_3 > 120°$.	P_3	P_1	P_1	P_2

THE MATHEMATICAL PUZZLE AS A STIMULUS TO INVESTIGATION.[1]

By WALTER B. CARVER, Cornell University.

The word investigation may be interpreted broadly enough to include all search for, and discovery of, truth which is new to the discoverer, regardless of whether it is or is not new to others. In this sense, a good high school student may be an investigator. Every teacher of mathematics has some scholars who have sufficient intellectual curiosity to ask themselves questions, and sufficient mathematical technique to discover the answers to some of these questions. All that is needed is something to arouse their interest and get them started. But the teacher, too, must be an investigator if he is to retain his enthusiasm and any freshness in his point of view. And making all due allowance for long teaching hours and heavy burdens of administrative work, the thing that primarily prevents most teachers from doing work of investigation is a certain inertia and lack of interest. As a pleasant and effective stimulus for student and teacher alike, the writer recommends the mathematical puzzle. If there is any merit in this recommendation, the best argument in its favor will be the presentation of a few examples.

From an excellent collection of mathematical puzzles by H. E. Dudeney[2] I quote the following:

Five ladies, accompanied by their daughters, bought cloth at the same shop. Each of the ten paid as many farthings per foot as she bought feet, and each mother spent 8s. 5¼d, more than her daughter. Mrs. Robinson spent 6s. more than Mrs. Evans, who spent about a quarter as much as Mrs. Jones. Mrs. Smith spent most of all. Mrs. Brown bought 21 yards more than Bessie, one of the girls. Annie bought 16 yards more than Mary and spent £3 8d. more than Emily. The Christian name of the other girl was Ada. Now, what was her surname?

Of course the point of the problem lies in the fact that the total number of farthings spent by each of the ten persons is a perfect square; and that we have the relation

$$x^2 - y^2 = 405,$$

where x^2 and y^2 represent respectively the number of farthings spent by any one of the mothers and her daughter. Since the farthing is the smallest English coin, we may assume that the values of x and y are to be integers, *i.e.*, that this is a Diophantine equation. We can find its solutions by expressing the right-hand member as the product of two factors, and equating these factors to $x + y$ and $x - y$ respectively. We thus find just five pairs of values,

$$x = 21,\ 27,\ 43,\ 69,\ 203,$$
$$y = \ 6,\ 18,\ 38,\ 66,\ 202,$$

[1] Read at the meeting of the Association, University of Rochester, Sept. 6, 1922.

[2] *Amusements in Mathematics*, London, 1917. The puzzle here quoted is number 140 in this collection. Dudeney conducts a puzzle page in the *Strand Magazine*. His first book *Canterbury Puzzles* is now in its second edition.

and the numerous statements in the puzzle enable us to identify these different solutions with the different mothers and daughters.

But any one with even a little mathematical curiosity will surely go further and ask himself questions. How many solutions can be found for the Diophantine equation

$$x^2 - y^2 = n,$$

where n is any given integer? Can the same methods be applied to the equation

$$x^2 - 5xy + 6y^2 = n;$$

or to any equation of the form

$$ax^2 + bxy + cy^2 = n,$$

where a, b, c, and n are any positive or negative integers? A whole field of mathematics is thus opened up, in which one may either investigate for himself or read the results of the investigations of others—or both.

The problem of finding right triangles whose sides are integers is a very old one; and it is well known that *all* positive integral solutions of the Diophantine equation

$$x^2 + y^2 = z^2$$

can be obtained from

$$x = (r^2 - s^2)t, \qquad y = 2rst, \qquad z = (r^2 + s^2)t,$$

r, s, and t being arbitrary positive integers. But less familiar is the problem of finding triangles whose sides are integers and one of whose angles is 120° or 60°, leading to the Diophantine equations

$$x^2 + xy + y^2 = z^2 \qquad \text{and} \qquad x^2 - xy + y^2 = z^2.$$

It is not difficult to show that

$$x = (r^2 - s^2)t, \qquad y = (2rs - r^2)t, \qquad z = (r^2 - rs + s^2)t$$

furnishes all solutions for the 120° case. The equation for the 60° case is rather more difficult. But one may see quite readily from a figure that if x, y, z are integral sides of a 120° triangle, then x, $x + y$, z and $x + y$, y, z are sides of two different 60° triangles; and that *all* solutions of the 60° triangle may be obtained in this way from the solutions of the 120° case. The smallest solutions are 5, 3, 7 for the 120° case, and 5, 8, 7 and 8, 3, 7 for the two corresponding 60° triangles.

A similar problem is that of finding a right circular cylinder and the frustum of a right circular cone of equal altitude and volume, the radii of the cylinder and of the upper and lower bases of the frustum being integers. If z is the radius of the cylinder, and x and y those of the bases of the frustum, we have

$$x^2 + xy + y^2 = 3z^2.$$

The simplest non-trivial solution of this Diophantine equation is 2, 11, 7. One can look for further particular solutions, or for some form that will give an

infinite number of solutions; or, most desirable of all, some form or forms which will give *all* solutions.

Consider another example.[1] Thirteen mice, one of which is white, are arranged in a circle. A cat, beginning with the white mouse as number one, counts around this circle up to a certain number n, and eats this nth mouse. Beginning with the next mouse beyond the victim as number one, and counting in the same direction, the cat again eats the nth mouse; and then repeats the process once more. What is the smallest value for n in order that the third mouse eaten may be the white mouse?

It is not difficult to see that we will have a value for n (not necessarily the smallest value) if we can find integral values for n, a, b, x, y, t, satisfying the following conditions:

$$\begin{aligned} n - 13x &= a, & n &> 0, \\ n - (13 - a) - 12y &= b, & 0 &< a < 14, \\ n - (12 - b) - 11t &= 1, & 0 &< b < 13. \end{aligned}$$

Eliminating a and b from the three equations, we have

$$3n = 13x + 12y + 11t + 26, \quad \text{or} \quad n = 4x + 4y + 4t + 9 + \frac{x - t - 1}{3}.$$

Putting

$$\frac{x - t - 1}{3} = z, \qquad \text{or} \qquad t = x - 3z - 1,$$

we have

$$n = 8x + 4y - 11z + 5, \quad a = -5x + 4y - 11z + 5, \quad b = 3x - 4y - 22z - 3,$$

and we thus obtain integral values satisfying the three equations by giving any arbitrary integral values to x, y, and z. It only remains to choose these values of x, y, and z in such a way as to satisfy the inequality conditions, namely

$$\begin{aligned} 8x + 4y - 11z + 5 &> 0, \\ 0 < -5x + 4y - 11z + 5 &< 14, \\ 0 < \quad 3x - 4y - 22z - 3 &< 13. \end{aligned}$$

Regarding x, y, and z as the rectangular coördinates of a point in space, one sees that the points satisfying these five inequalities lie within a portion of space below the plane $8x + 4y - 11z + 5 = 0$ and bounded by a prismatic surface whose cross-sections are parallelograms. Considering now only the lattice of points whose coördinates are integers, we wish to find the point of this lattice lying inside the prismatic surface and at the least distance below the plane $8x + 4y - 11z + 5 = 0$. One easily deduces that z must be less than 1. For $z = 0$, the inequalities in x and y can not be satisfied with integers; but for $z = -1$ we have a number of solutions. In particular, $x = 7$, $y = 7$, $z = -1$ gives $n = 100$, the smallest value for n. Some other values for n are 104, 112, 116, 124, 128, 136, 140, 144, 148, 152, etc. If one finds thus all values of n up to

[1] This is the third part of Dudeney's puzzle number 232.

1716 (= 11 × 12 × 13), all further values can be obtained by adding multiples of 1716 to these.

For a final example, I once more quote Dudeney.[1]

Eight men had been dining not wisely but too well at a certain London restaurant. They were the last to leave, but not one man was in a condition to identify his own hat. Now, considering that they took their hats at random, what are the chances that every man took a hat that did not belong to him?[2]

Let K_n represent the number of permutations of n letters such that no letter goes into itself. One sees readily that $K_1 = 0$, $K_2 = 1$, $K_3 = 2$. If one defines, for convenience, $K_0 = 1$, the formula

$$K_n = n! - (C_n^1 K_{n-1} + C_n^2 K_{n-2} + \cdots + C_n^{n-1} K_1 + C_n^n K_0)$$

follows from the fact that the successive terms in the parentheses give respectively the number of permutations sending just one letter, just two letters, . . . just $(n - 1)$ letters, all n letters, into themselves. By means of this formula we readily deduce

$$K_4 = 9, \quad K_5 = 44, \quad K_6 = 265, \quad K_7 = 1854, \quad K_8 = 14833.$$

Hence the chance that each of eight men should get the wrong hat is 14833/8! Two recursion formulas, simpler than the one above but rather more difficult to establish, are

$$K_n = (n - 1)(K_{n-1} + K_{n-2}) \qquad \text{and} \qquad K_n = nK_{n-1} + (-1)^n.$$

The following interesting relations were pointed out to the writer by Professor W. A. Hurwitz:

$$K_n = \int_0^\infty (t - 1)^n e^{-t} dt,$$

$$\frac{e^x}{1 - x} = K_0 + K_1 x + \frac{K_2}{2!} x^2 + \frac{K_3}{3!} x^3 + \cdots + \frac{K_n}{n!} x^n + \cdots,$$

$$K_n = n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\right).$$

This last relation shows that as n increases, the chance that each of n men should get the wrong hat approaches $1/e$.

These examples are perhaps sufficient to indicate the rather serious sort of mathematics into which one may be led by following up apparently trivial or foolish puzzles.

[1] Number 267 in his Amusements.

[2] This problem is discussed by Lucas in his *Théorie des Nombres*, chapter XIII; and references are there given to other treatments by Euler and Neuberg.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

QUESTIONS.

The following questions are proposed by Professor Harris Hancock, the first being suggested by the replies to Question **36**, 1 (*1922*, 255–7).

48. In what quadratic realms of rationality and for what values of p (which is a prime integer) is the function $\frac{x^p - 1}{x - 1}$ factorable?

For example,

$$\frac{x^5 - 1}{x - 1} = \left(x^2 + \frac{1 + \sqrt{5}}{2}x + 1\right)\left(x^2 + \frac{1 - \sqrt{5}}{2}x + 1\right)$$

is evidently factorable in the realm $R(\sqrt{5})$; while $\frac{x^7 - 1}{x-1}$ has the factors $(x^3 - ax^2 + bx - 1) \times (x^3 - bx^2 + ax - 1)$, where $a = (-1 + \sqrt{-7})/2$, $b = (-1 - \sqrt{-7})/2$, and is therefore factorable in the realm $R(\sqrt{-7})$.

49. What is the most general value of the constant T for which the differential equation

$$\frac{d^2y}{dx^2} + \frac{ay + T}{(by^2 + cy + dx^2 + ex + f)^2} = 0$$

admits solution in the form of elliptic functions, a, b, c, d, e, and f being constants?

A special case is given by Euler, *Institutiones Calculi Integralis*, p. 153.

DISCUSSIONS.

Professor R. M. Mathews quotes a construction for a certain type of circular cubic, and discusses an exceptional curve for which the construction fails.

Mr. Philip Fitch makes the suggestion that less emphasis be placed on the problem of orthogonal trajectories, and that the more general problem be studied in which two families of curves cut one another, not at right angles, but in such a way that the slopes at points of intersection are related according to any given law. Teachers of differential equations are probably glad enough so to vary a standard problem, provided that a sufficient supply of interesting examples is available to illustrate what they are doing.

We are from time to time reminded of some inconsistency or inexactitude of expression that pervades some special branch of mathematics; but we probably agree that the general aim of mathematical writers is to exhibit extreme precision and carefulness in the use of language. The note contributed by Professor A. A. Bennett shows how easily the writer of an elementary text fails to reach this ideal. No doubt a number of the flaws to which allusion is made are the result, not so much of inexact thinking as of a desire to use the simplest possible language. Certainly this would account for the frequent use by some authors of "greater than" where the meaning is "greater than or equal to" or "numerically greater than"; though the simplicity so attained is never anything but illusory. Professor Bennett does not select his authors for the multitude of their errors. Had he chosen one of the real offenders, he might still be asking "Why?" However, let us not overlook the fact that the bad usages are fast

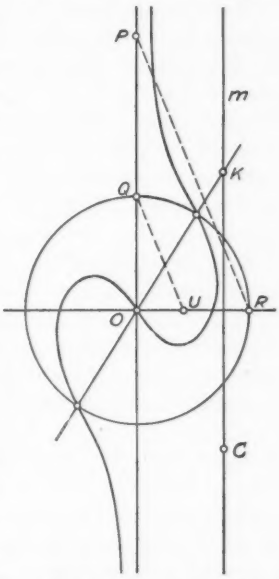
becoming obsolete, and that there is a strong movement today, of which the present paper is a sign, towards exactitude in mathematical phraseology.

I. An Exceptional Circular Cubic.

By R. M. Mathews, Wesleyan University.

It is well known that the locus of the intersection of any member of a pencil of coaxial circles with its diameter through a fixed point F is a circular cubic. The point F lies on the curve and is the *singular focus*, that is, the intersection of the tangents at the circular points at infinity. The line of centers of the circles is called the median of the curve. These curves have been studied synthetically by Schroeter, Durège and Lagrange.[1]

The converse theorem may be stated: any circular cubic which contains its singular focus F may be considered as the locus of the intersection of any member of a coaxial pencil of circles with its diameter through F. This theorem has been given by the authors named and by Loria and Hilton.[2] The object of this note is to examine a special cubic for which the diameter becomes indeterminate, and to arrive at a definite construction for this case.



The equation

$$(x^2 + y^2)x - 2gx - 2fy = 0 \qquad (1)$$

defines a proper circular cubic which is symmetric with respect to the origin O. The asymptotes are the y axis and the isotropic lines at O; so O is at once singular focus and principal point—intersection of the curve with the real asymptote. The circles of the pencil are concentric at O and so form a coaxial pencil bitangent at the circular points at infinity. The theorem is true but the simple correlation of circle and diameter is lost.

The correlation may be effected as follows. The equation (1) may be replaced by the system parametric in r

$$x^2 + y^2 = r^2, \quad (2) \qquad y = \frac{r^2 - 2g}{2f}x. \quad (3)$$

Plot the point $C = (2f, -2g)$ and through it draw m parallel to the y axis. On Ox take $OU = +1$. A circle of radius r cuts Ox at R and Oy at Q. Draw RP

[1] H. Schroeter, *Mathematische Annalen*, vol. 5, 1872, pp. 50–82; H. Durège, *ibid.*; A. Lagrange, *Nouvelles Annales de Mathématiques*, series 3, vol. 19, 1900, pp. 66–74.

[2] G. Loria, *Spezielle algebraische und transzendente ebene Kurven*, vol. 1, Leipzig, 1910, p. 35. H. Hilton, *Plane algebraic curves*, Oxford, 1920, p. 226, Ex. 28.

parallel to UQ to cut Oy at P. On m take $CK = OP$ and in the same direction. Then OK is the required line.

II. On Trajectories in General.

By Philip Fitch, Denver, Colo.

The treatment of trajectories in the better known works on differential equations is confined principally to those involving a given angle and more particularly to orthogonal trajectories. There are, however, numerous others of a different type, and having a practical value, that receive no mention.

Because of the common property of most of them, namely—that they always cut all the curves of the original family—the term "transversals" might be more appropriate.

If we let $f(x, y, \alpha) = 0$ represent the equation of a certain family of curves, we may have $\phi(x, y, dy/dx) = 0$ represent the differential equation of the same family. And in the expression $\phi(x, y, dy/dx)$ we may substitute a function of dy/dx, say $\psi(dy/dx)$, and obtain a differential equation of a new family of curves that will have some relation to the original family. Now the nature of the trajectory or transversal depends wholly upon the $\psi(dy/dx)$ and its relation and usefulness are determined by this function.

For example, take the case of bodies or groups of bodies moving in paths that are curves belonging to the same family, it might be necessary to obtain the locus of points where these bodies would possess a common property. This property could be equal momenta, equal kinetic energies, velocities in the same direction, and others, all of which may be represented as functions of dy/dx.

The value of these problems from a practical as well as an educational and purely mathematical standpoint cannot be over-emphasized.

Among the problems of this type that have been found useful is that of determining the transversals for velocity in the same direction of a series of streams of water shot out horizontally from the same level under different pressures or heads. Since the paths of these streams are parabolas with vertices and axes in common, we find the transversals to be a family of straight lines, concurrent at the point of efflux of the streams.

Another illustration is the determination of the transversals for equal kinetic energies for the streams mentioned above. In this case the transversals are a certain family of transcendental curves.

III. Why?

By Albert A. Bennett, University of Texas.

Attention has been called in this Monthly [1] to the loose attempts made to define "complex," "imaginary," "pure imaginary," as applied to numbers, in most American text-books in elementary algebra. There are certain other wide-

[1] E. S. Allen, "Definitions of imaginary and complex numbers" (*1922*, 301–303).

spread practices to which attention might also be called. For the sake of being specific, the excellent and widely-used text by Rietz and Crathorne will be mentioned, and page references will be to the "Revised Edition," 1919, of their *College Algebra*. The use of references of this sort will condense the discussion, but the writer realizes that this book may be more than usually free from lapses of the form cited, if these be indeed lapses.

P. 30, 8th line. "The square root of a negative number." One is at a loss to know whether even for a positive number, the authors would have one speak of "*the* square root," since the question of signs remains somewhat ambiguous, but surely no convention has been adopted which will justify the use of the definite article in the case of square roots of negative numbers. Why not agree that $\sqrt{n}$ is defined for non-negative real numbers *only*, and that the two square roots of $-n$, where n is positive, will be denoted by $\sqrt{n}\,i$ and $-\sqrt{n}\,i$, respectively?

P. 48, Exercise 4. Why should the illustrative example, at the bottom of page 46, be made an exercise for the student, two pages later? Is this merely to determine whether the student has read the text, or is it an accident due to the authors' not having read the exercise?

P. 58, Exercise 24. Why should a problem asking for positive integers be recorded as having an answer equal to zero?

P. 67, II. Why should the authors remark "since $r \neq 0$," when this was not given in the hypothesis, and since in the first application of the theorem, namely in III at the foot of the page, the case $r = 0$ is required?

P. 67, II. Why state the theorem, "a quadratic equation has two roots arithmetically equal but opposite in sign when and only when the term in x vanishes," as though a quadratic equation could not be in y or z instead of x? Why speak of the "known term," for a quadratic in x, meaning the term independent of x, when in the first application this "known term" contains an unknown, k, which is to be determined subject to certain conditions? Many books use the phrase "constant term" in an equation in which the unknown is at least constant, or "unknown" for a function in which the x although a variable is assuredly "known."

P. 71, Exercises. In plotting the graphs of quadratic functions, why are the exercises given in the form of quadratic functions *equated to zero?* Being given equations only, why cannot the student multiply by an arbitrary non-vanishing constant at will, and so obtain other graphs? If this is desired, why is it not explained?

P. 86, 1st line. Why are inequalities defined at this late stage after use has been made of them, P. 3, line 8; P. 13, line 1; P. 23, line 7; P. 25, next to bottom line; P. 68, 7th line from bottom; etc.?

P. 90, 4th line from bottom. Why must the "and so on" be made to reappear in each proof by mathematical induction, instead of being included in the notion of positive integers, where it belongs? Why not conduct the discussion somewhat as follows?—From the character of positive integers, we shall assume (and it may be proved) that if a theorem is true of a positive integer there must be a

first positive integer for which the theorem holds. Suppose if possible that the given proposition which we have established in the particular case $n = 1$ should fail for some positive integer. Then it must fail to be valid for a certain first one which we shall call $r + 1$. Here $r + 1$ is a positive integer at least as great as 2. Hence r is a positive integer, and one for which the proposition is valid. The argument is then carried out to show that the truth for r implies the truth for $r + 1$. Our hypothesis that the proposition fails to be valid is therefore seen to lead to an inconsistency and, unless other logical errors have been introduced, the proposition is therefore proved to be valid for all positive integers.

In this connection why not distinguish boldly between psychological induction which gains plausibility by the mere multiplicity of valid cases, and logical induction? To establish the theorem directly for $n = 2$, and $n = 3$, after the case $n = 1$ is proved, merely delays the argument. If the student finds it worth while even to attempt to prove the theorem, the stage at which psychological induction has value has been already passed, and the case $n = 1$ affords an ample start. If the proposition has a meaning for $n = 0$, this case is usually even simpler to demonstrate, and although 0 is not a positive integer, it suffices to start the induction.

Perhaps geometry is too elusive to permit of rigorous elementary text-books, but why must algebra books abound in opportunities for queries of this sort?

RECENT PUBLICATIONS.

REVIEWS.

Space—Time—Matter. By HERMANN WEYL. Translated from the German by Henry L. Brose. New York, E. P. Dutton and Co., 1922. 330 + xi pages. Price $7.50.

Professor Weyl's book was first published in 1918 and immediately achieved a remarkable success, the fourth edition being published in 1920. It is this latest edition which Mr. Brose, who has already successfully translated Einstein's own popular account of the theory of relativity, presents to the English-speaking scientific public. There is no indication of any collaboration with the author and indeed it is not clear whether the translation was "authorized" or not. This may account for many faults, some of which are indicated in detail below. It is claimed that "great care has been taken to render the mathematical text as perfect as possible." The number of misprints is, however, lamentably great and their occurrence makes the book very difficult reading for those to whom the translation can be of any possible use. The printing involved is obviously difficult, but the price asked for the book should certainly warrant better success. In a symbolism where the position of a label (above or below a symbol) is all important it is fatal for the printer to exhibit a certain carelessness in this respect.

Professor Weyl's book has been favorably reviewed by many competent

authorities.[1] It is by no means adapted for a cursory reading but demands leisurely study. And it will be of inestimable value to those who "think it out" for themselves. For the mathematician especially, the first chapter with its account of affine geometry will be very interesting. This subject is receiving considerable attention just now and Weyl deserves the credit of focussing this attention.[2] An account of Riemann's concept of a metrical space and of Weyl's own extension is given in Chapter 2. The contrast between the "action at a distance" character of Euclidean space and the "action through a medium" character of Riemann space is emphasized. The final chapters of the book deal with Einstein's idea of the relativity of space and time. In the discussion of the fundamental problem of the single statical gravitating center (p. 252) it is not clear what is meant by "the distance $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ from the centre." Some appeal from Riemann to the ideas of Pythagoras is surely being made here. But it was on reading § 35 on "The metrical structure of the world as the origin of electromagnetic phenomena" that we were finally convinced that the author belongs to that school of teachers which deprecates the lucid, orderly lecture on the ground that it is too easy and does not exercise sufficiently the mind of the reader. When we were dejected by our failure to understand a particular passage the ill-natured remark of Voltaire was recalled; to the effect that when two Germans are discussing a subject in such a manner that neither understands the other, they are talking metaphysics; but when they rise to such heights that neither understands himself, they are talking higher metaphysics. We may well be consoled, however, by the fact that the great German mathematicians—Hilbert, for example—are always clear and understandable. In making these remarks it will be clear that our strictures are meant not for what Weyl has to say but for his manner of saying it. He is, next to Einstein, the most serious contributor to the theory of relativity.

We add a list of errata and remarks for the non-expert reader of the book.

P. 21, equation (4) e^k should read e_k. It is unfortunate that upper (contravariant) labels were not used throughout for the coördinates and their differentials instead of the lower (covariant) labels. If this were done, the elegant property of the symbolism by which a summation (dummy) label occurs once above and once below in a term would be preserved. Equation (4) would read $e'^i = \Sigma \alpha_k^i e^k$.

P. 22, line 3 from top; $\overrightarrow{A'B'} = a' - b'$ should read $\overrightarrow{A'B'} = a'$.

P. 23, lines 4 and 2 from bottom; "quantum" should read "quantity."

P. 36, formula (23)′; for $\bar{\xi}^l$ read $\bar{\xi}_l$.

P. 52, line 13 from top; for $\bar{\xi}^r$ (at end of line) read $\bar{\xi}_r$.

P. 60, formula 41; for $\frac{\partial u^i}{\partial x_k}$ read $\frac{\partial u_i}{\partial x_k}$.

P. 87, at bottom; the remark that "the equation

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2$$

[1] See *Bulletin of the American Mathematical Society*, vol. 28, 1922, p. 215, for a review by G. D. Birkhoff, and *Nature*, vol. 109, 1922, p. 634, for a review by A. S. Eddington.

[2] An interesting paper on the same subject is that of W. Blaschke in *Mathematische Zeitschrift*, vol. 12, 1922, p. 262.

will hold more exactly the smaller du_1 and du_2 are taken" will grate on the ears of those accustomed to the Leibniz treatment of differentials.

P. 90, line 4 from top; for "total" read "exact."

P. 96, The remark, line 3 from top, is somewhat misleading. Riemann's definition of curvature is not an extension of Gauss' but a different definition altogether. One cannot be too careful in using the word "curvature" in these days when non-mathematicians talk so glibly about the curvature of space.

P. 96, at middle; for $(\Sigma x_i dx)^2$ read $(\Sigma x_i dx_i)^2$.

P. 107, line 8 from bottom; One must pillory the remark that "an infinitesimal element of surface is only a part (or more correctly, the limiting value of the part) of an arbitrarily small but finitely extended surface."

P. 108, line 2 from bottom; for ∂ read δ.

P. 112, line 5 from top; for $w_{ik}f_i$ read $w^{ik}f_i$. The justification for the formula immediately following is not clear.

P. 115, line 2 from top: The sneer in the words "This will relieve the minds of those who disapprove of operations with differentials" is not justifiable after the remark on page 107 (see above).

P. 136, line 12 from top; for e read ϵ.

P. 141, last line; for $d\gamma^i$ read $d\gamma_k{}^i$.

P. 161, line 12 from bottom; for "statistical" read "statical."

P. 187, line 10 from bottom; Voltaire would undoubtedly dignify by the term *high metaphysics* the statement "Suppose we have two twin brothers who take leave from one another at a world point A and suppose one remains at home . . . whilst the other sets out on voyages. . . . When the wanderer returns home in later years he will appear appreciably younger than the one who stayed at home."

P. 196, line 14 from top; for "Eichwald" read "Eichenwald."

P. 202, lines 4 and 5 from top; for $\frac{t}{dt}$ read $\frac{d}{dt}$.

P. 211, line 4 from bottom; for "deducted" read "deduced."

P. 231, in right hand side of equation of line 2; for u_k read u^k.

P. 258, lines 5 and 7; replace the minus by equality signs.

P. 262, last line; for u^i read u_i.

In conclusion we may explain why we have directed attention to the faults rather than to the merits of this translation. The widespread interest in the theory of relativity is a rare phenomenon. Not often is such an interest in the methods and results of mathematical analysis displayed by men cultured in other departments of learning. To those who believe in the importance of these methods for the progress of civilization it must seem particularly unfortunate if this curiosity is stifled and distrust aroused by difficulties due to the mode of presentation of the theory; those inherent to it are, we all know, sufficient.

FRANCIS D. MURNAGHAN.

Cours Complet de Mathématiques Spéciales, Volume 3, Mécanique. By J. HAAG. Paris, Gauthier-Villars et Cie. 1922. 8vo. viii + 188 pages.

This brief volume, although on the whole rather elementary, has enough individuality to make it interesting reading. In the opening chapters, which deal with kinematics, the motion is frequently defined by giving the relationship between displacement and time, rather than by giving the differential equation which defines such a relationship. This has certain advantages, since the differential equation naturally arises in dynamics and its solution can be taken up at that time. Considering the general scope of the book, it is surprising to encounter a Fourier series in connection with the discussion of vibratory motion.

The section on dynamics deals mainly with the dynamics of a particle. There is a chapter of fourteen pages entitled "Notions on the Dynamics of Systems" and a final chapter on "The Units of Mechanics." As there are no numerical exercises, there is no objection to this arrangement. The last two chapters deal with statics. Rolling friction, sliding friction, pivot friction and journal friction are discussed and a table of coefficients of friction for a number of different substances is also given.

Anyone using the book with a class would probably supplement it with a number of exercises.

PETER FIELD.

Leçons sur le Problème de Pfaff. By EDOUARD GOURSAT. Paris, J. Hermann. 1922. Price 30 francs.

In this book Goursat has produced what is probably the most scholarly work on the subject. The treatment is thoroughly up to date and exact references to original sources are more abundant that is usual in such treatises. In order to understand the book one must have freshly in mind the fundamental theorems relative to partial differential equations. Goursat makes frequent reference to his own work, *Leçons sur l'integration des équations aux dérivées partielles du premier ordre*, and seems to regard the present book as a sort of continuation of it.

The first two chapters deal with the "Problem of Pfaff" properly so called, particular emphasis being put on the properties of bilinear covariants. In the next two chapters a study is made of symbolic differential forms and of their application to the "Problem of Pfaff." In the fifth chapter there follows a study of integral invariants. The last three chapters are devoted to the most recent progress in the subject, particularly to that due to Cartan to whose work frequent reference is made.

The typography, etc., are good although a few misprints were noticed. The lack of binding and the uncut pages common in French books as they come from the press always annoy an American reader.

TOMLINSON FORT.

Elementary Calculus. By F. S. WOODS and F. H. BAILEY. Boston, Ginn and Company, 1922. 8 + 318 pages. Price $3.00.

As the preface states, this book is intended for first-year college students. In the light of this fact, it is not surprising that the treatment of the subject is occasionally simplified at the expense of rigor; for example, Taylor's and Maclaurin's series are discussed without mention of convergence or the remainder term. The explanations are, with few exceptions, very clear and easy to understand, and the abundance of problems which has come to be associated with the names of Woods and Bailey will add much to the usefulness of the book. In view of the fact that the book is primarily intended for engineering students, the introduction of the derivative through the concepts of velocity and accelera-

tion is a welcome departure from the geometrical presentation in terms of slope and curvature, which in this text is taken up later.

The following criticisms of a specific character may be made:

In the definition of a limit on page 2 the authors state that the difference between the variable and its limit "becomes and remains less than any small quantity which may be assigned," evidently meaning "any small positive quantity."

In connection with integration it might well have been proved that any two indefinite integrals of a function differ by a constant.

On page 5 appears the logical error of dividing $\frac{1}{2}$ mile by $\frac{1}{60}$ hour.

The discussion of pressure on page 68 calls attention to the confusion which appears in many books between the two meanings which are attached to the word "pressure," namely, *total force* and *force per unit area.*

In the proof, on page 120, that

$$\lim_{h \to 0} \left(\frac{\sin h}{h} \right) = 1$$

it will probably not be clear to the student that $BD + DB' > BAB'$.

The definitions of the inverse trigonometric functions, on page 130, as multiple-valued functions lead to much unnecessary complication in the formulæ for their derivatives.

The use of the term "linear acceleration," on page 135, to denote the tangential component, d^2s/dt^2, of the acceleration in plane curvilinear motion conflicts with the customary usage and, in particular, with the common expression of Newton's second law of motion, namely, $f = ma$, where f, m, and a are the measures of force, mass, and acceleration respectively, in terms of suitably chosen units.

It is difficult to understand the reason for introducing, as is done on p. 181, the notation $(df/dx)_y$ for $\partial f/\partial x$, and other analogous notations, and making practically no use of them later.

On the whole, many of the subjects treated seem a bit beyond the comprehension of the average freshman, but the book would probably be very satisfactory with sufficiently mature students.

HORACE L. OLSON.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

THE NEWTONIAN CLUB OF THE UNIVERSITY OF ALABAMA, University, Ala.

Seventeen students, with the coöperation of Professor Fort, Professor Lewis, and Mr. Dahlene, organized the Newtonian Club on September 13, 1921, to "*add* to the interest in mathematics, *subtract* from its apprehensive terror, *multiply* its benefits, and *divide* its snares." They elected as president, F. L. Davis, student-assistant in mathematics; vice-president, Reese Malette; secretary-treasurer, Louise Sandifer. Membership is limited to twenty including both active and

honorary members. Active members are elected by a two-thirds vote from students who have shown proficiency in analytical geometry and differential calculus; honorary members are the faculty members. The club meets on the first and third Tuesday nights in each month, generally at the family or fraternity homes of its members. In 1921–1922 the lives of Newton, Leibnitz, Napier, Einstein, and women mathematicians were studied; also such topics as the invention of the calculus and of logarithms, and the Einstein Theory. At one meeting a concert given in Philadelphia was heard by radio. For the year 1922–1923 the following officers were elected: President, Ramsey Reed '23; secretary-treasurer, Jesse Williams '23; program committee, the officers and Professor Fort.

September 12, 1922: Organization and election of new members.

October 10: "A type of algebraic correspondence" by Professor I. A. Lewis.

November 14: "Reasons for studying mathematics" by Professor Tomlinson Fort; "Circles connected with a triangle" by Harold Friedman '23.

December 12: "Determinants" by Esther Frank '25, John Steiner '23, and S. Rubrai '23.

(Reported by Miss Sandifer and Professor Fort.)

The Mathematics Club of Iowa State College of Agriculture and Mechanic Arts, Ames, Ia.

The Mathematics Club of Iowa State College was organized in October 1920 with a membership of fifteen including junior, senior, and graduate students majoring in mathematics. The purpose of the club is to stimulate interest in mathematics outside the usual text books and to promote sociability between students and faculty. The following officers were elected for the year 1920–1921: President, Ruth Dewey, Gr.; secretary-treasurer, Lillian Willson 22; faculty adviser, Professor Marian Daniells.

The program for the first year included papers on "Mathematical fallacies," "Mathematics of life insurance," "Mathematicians past and present" and a discussion of the Newton-Leibniz controversy. The final meeting of the year was a dinner at which were served non-isotropic ellipsoids, stuffed paraboloids, permutations and combinations and similar dainties. The program which followed the dinner included the play "The Flatlanders" and vaudeville by the Mathematics Department quartet. The first meeting of the year 1921–1922 was a picnic. At the October meeting the following officers were elected: President, Mary Battell '22; secretary-treasurer, Gilbert Witmer,—who was succeeded in the winter quarter by Alberta Wolfe, Gr.; faculty adviser, Professor Daniells.

November 2, 1921: "Why study mathematics" by Professor E. R. Smith.

December 7: "Mathematics and chemistry" by Professor E. I. Fulmer, of the Department of Chemistry.

January 4, 1922: "History of the development of logarithms" by Thelma Tollefson '23; "Life of Edward Wright" by O. A. Bock '23.

February 1: "Teaching of high-school mathematics" by Professor Gertrude Heer.

March 1: "The game of Nim" by A. L. Young, Gr.

April 5: "The planimeter and integration by mechanical means" by Mary Battell '22; "The abacus and other forms of counting machines" by Alberta Wolfe, Gr.

May 3: "Mathematics and music" by Florence Catlin, Gr.

In May the faculty of the department entertained at dinner the members and prospective members of the club. The dinner was followed by a program of music and toasts on the following subjects: "The integrating factor," "Essential singularities," "Harmonic progressions," "Absolute values," "A plane talk," and "Reductio ad absurdum" consisting of episodes from *Alice in Wonderland* and *Through the Looking Glass*.

(Reported by Professor Daniells.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3011. Proposed by E. T. BELL, University of Washington.

In a certain paper it is stated that "it is easy to prove that, if $p > 0$ is an integer, the relation $a_1 \sin \frac{\pi}{2p} + a_2 \sin \frac{2\pi}{2p} + \cdots + a_{p-1} \sin \frac{(p-1)\pi}{2p} + a_p = 0$ necessitates $a_1 = a_2 = \cdots = a_{p-1} = a_p = 0$, the a's being integers." Prove it.

3012. Proposed by R. A. JOHNSON, St. Paul, Minn.

It is well known that, of all polygons with given sides, that one which can be inscribed in a circle has maximum area. Show how to construct a polygon with its vertices concyclic and with its sides in order equal to the sides of a given polygon.

3013. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that $\lim_{n \to \infty} \sum_{r=0}^{r=m-1} \frac{4m}{4m^2 + (2r+1)^2} = \frac{\pi}{4}$.

3014. Proposed by the late T. M. BLAKSLEE, Ames, Iowa.

If 1, p, d are the radius and sides of the regular inscribed pentagon and decagon, and if a triangle be formed from these three lengths, determine the angle opposite p without the use of trigonometric tables.

3015. Proposed by J. G. COFFIN, New York City.

A heavy particle at the end of a weightless, flexible, inextensible string is set in motion. The string is attached to the side of a cylindrical post of radius 7 and the initial motion is such that, when the string is tangent to the post, the particle is moving as a conical pendulum. What is the curve made by the string on the post? What is the curve in space described by the particle? What time is required to wind up completely? As an interesting variation assume that the string is extensible.

3016. Proposed by R. B. ROBBINS, University of Michigan.

Solve the following recurrence relations, expressing the solution in terms of initial values U_1 and M_1 and the variable n which is always a positive integer:

$$2M_n = 2K_nU_n = M_{n-1} + U_{n-1}, \qquad K_n = M_{n-1} + U_{n-1} - M_{n-1}U_{n-1}.$$

SOLUTIONS.

2932 [1921, 467]. Proposed by R. C. ARCHIBALD, Brown University.

De Ville gave in 1629 the following construction for an approximation to the side of a regular polygon of n sides inscribed in a given circle: On a diameter AB construct an equilateral triangle ABC; divide AB into n equal parts, join C to D, the first point of division of AB, and let CD produced intersect the circumference in E; then the chord of the arc AF, the double of AE, will be approximately equal to the required side of the polygon. Construct a table of values for $n = 5$ to $n = 20$ exhibiting the extent, in minutes and seconds, to which the above construction

is in error in connection with the central angle subtended by the constructed side of the polygon. [Somewhat similar tables have been made for constructions by Bosse and Bernard.]

Solution by H. S. Uhler, Yale University.

Take a system of rectangular axes in such a position as to make the coördinates of A, B, C respectively $(a, 0)$, $(-a, 0)$, and $(0, -a\sqrt{3})$. Then $\overline{DA} = 2a/n$ and the coördinates of D are $(a - 2a/n, 0)$.

The equation of the line CDE is

$$y + a\sqrt{3} = \frac{\sqrt{3}}{1 - \frac{2}{n}}\, x.$$

Making the last equation simultaneous with that of the circle

$$x^2 + y^2 = a^2$$

the ratio of the ordinate to the abscissa of E is found at once to be

$$\tan \theta_n = \frac{\sqrt{3}\left[\sqrt{3 - 2\left(1 - \frac{2}{n}\right)^2} - 1\right]}{2\left(1 - \frac{2}{n}\right)},$$

where $\theta_n = \angle\, AOE$ = one half of the central angle subtended by the chord $\overline{AF}$.

Using Barlow's tables for the roots, and Bauschinger and Peters' eight place logarithmic tables I calculated the data tabulated below. All the numbers in the middle columns were verified by independent computation involving the corresponding formula for $\sin \theta_n$.

n	$\theta_n - 360°/n$		=	$(error)''$
5	0°	42′	41″.8	2561.8
6	0	56	48.7	3408.7
7	1	5	53.7	3953.7
8	1	11	25.8	4285.8
9	1	14	33.4	4473.4
10	1	16	3.8	4563.8
11	1	16	29·1	4589.1
12	1	16	10·7	4570.7
13	1	15	23·1	4523.1
14	1	14	16·3	4456.3
15	1	12	57.1	4377.1
16	1	11	30.2	4290.2
17	1	9	59.0	4199.0
18	1	8	25.7	4105.7
19	1	6	52.1	4012.1
20	1	5	19·1	3919.1

Also solved by W. C. Eells.

2933 [1921, 467].

Dudeney's[1] Problem, 1902: With ruler and compasses only, divide an equilateral triangle into four rectilinear pieces which may be put together so as to form a square.

Solution by H. C. Bradley, Massachusetts Institute of Technology.

Let ABC be the given triangle, D and E the middle points of the sides AB and BC. Construct S, the side of a square equal in area to the given triangle. This may be found by ruler and compasses since it is the geometric mean between AD and the perpendicular from C on AB. Describe a circle with center D and radius S cutting AC in F; draw the straight line DF and take

[1] Dudeney gave without proof in *Canterbury Puzzles* a solution which he presented to the Royal Society. Editor.

G on FA so that FG is one half CA. If FG is not greater than FA, or what is the same thing, if F falls on the segment CM where M is the middle point of CA, the construction is possible. Drop perpendiculars from E and G upon FD with feet at J and H, respectively. The figure will then be divided into four pieces, $DBEJ$, $HGAD_1$, CFJ_1E_1, $G_1H_1F_1$, where $D_1 \equiv D$, $J_1 \equiv J$, etc. The first three may be reassembled by putting D_1A and DB, EB and E_1C together. Then FC and AG will lie in a straight line, since the angles at A, B and C form 180°. Then the fourth piece may be placed with F_1G_1 in coincidence with FCG, since $FC + AG = FG$ by construction. The resulting figure is a rectangle for the adjacent angles at E, E_1 are supplementary; similarly for the angles at F, F_1, D, D_1 and G, G_1. Also the angles at H, H_1, J_1, J are right. It may be shown that $JD + D_1H = S$ and this suffices to show that the rectangle is a square.

Since no use has been made of the fact that the triangle is equilateral, the above process of dissection applies to any triangle in which certain conditions, seen from the above, are satisfied.

If a triangle is such that its altitude is equal to its base and neither of the two base angles exceeds 90°, the four pieces may be cut out by two straight cuts only. Let AC be the base, the altitude $BD = AC$, and E, F the middle points of AB and BC, respectively. Project E and F upon AC in J and K, respectively; then $EJKF$ is a square and it may be easily shown that the two cuts along the diagonals EK and JF give the required four pieces.

NOTE BY OTTO DUNKEL, Washington University.

Proof that the rectangle is a square. Since in an equilateral triangle $CD > AD$, the geometric mean S must satisfy the inequality $CD > S > AD$. Now $DM = AD$ and hence M lies within the circle of radius S and center D and C lies without. Hence F lies within the segment CM and G within MA.

Project C, A, B upon FD in the points C', A', B'. Then $C'F + HA' = FH$ from the construction for G; $C'J = JB'$; and $A'D = DB'$. Subtracting the first equation from the second there results $FJ - HA' = JB' - FH$ or $FJ + FH = JB' + HA' = JD + DB' + HD - A'D = JD + HD$. On the other hand we have $(FJ + FH) + (JD + HD) = 2S$ and, since it has been shown that the two parentheses are equal, it follows that $(FJ + FH) = (JD + HD) = S$.

Also solved by ARTHUR PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Miss CONSTANCE WIENER, formerly a graduate student at the University of Chicago, has been appointed instructor of mathematics at Smith College.

At Wesleyan University, Middletown, Conn., Mr. H. E. ARNOLD and Mr. G. W. BAIN have been appointed instructors of mathematics. Professor B. H. CAMP has been granted leave of absence for the year 1923–1924.

Dr. V. E. POUND, of Toronto University, has been appointed instructor of mathematics at the University of Buffalo.

At Union College, Mr. W. L. WARNER has resigned his instructorship of mathematics, and Mr. A. P. J. BOUDREAU has been appointed instructor.

At Lafayette College, Mr. R. J. W. TEMPLIN has resigned his instructorship, and Mr. J. A. BENNER, of Pennsylvania State College, has been appointed instructor of mathematics.

Mr. P. R. YODER, of the University of Kansas, has been appointed professor of mathematics and physics at Blue Ridge College, New Windsor, Md.

Miss HARRIET L. HASKINS, a graduate of Vassar College, has been appointed to teach mathematics and science at Eastern College, Manassas, Va.

Mrs. ANNETTE HIGHSMITH, of Peabody College, has been appointed instructor of mathematics at Meredith College for the year 1922–1923.

At Hampden-Sidney College, Professor J. S. MILLER has resigned to accept a position at Emory and Henry College. Mr. MACON REED, of Columbia University, has been appointed head of the department of mathematics, and Mr. C. D. LAWS, of the University of Georgia, an instructor.

At Greenville (S. C.) Woman's College, Miss ISABEL HARRIS, of Columbia University, served as professor for the academic year 1921–1922. Miss ROSE B. WOOD, of Barnard College, is professor of mathematics at present, and Miss MARY L. GAMBRELL has been instructor since the fall of 1921.

At the University of South Carolina, Mr. M. A. HILL and Mr. C. C. EDWARDS have been appointed instructors of mathematics.

At the Georgia School of Technology, Professor FLOYD FIELD has been elected Dean of Men. Mr. R. M. MUNDORFF has been appointed assistant professor and Mr. H. K. FULMER and Mr. E. R. C. MILES have been appointed instructors of mathematics. Mr. W. H. BOERCKEL resigned his instructorship in June, 1922, to accept a position in the high schools of Philadelphia.

Mr. R. G. DEMAREE, of the University of Chicago, has been appointed head of the department of mathematics at Kentucky Wesleyan College, Winchester, Ky.

At Maryville College, Maryville, Tenn., Professor J. A. HYDEN has been granted a year's leave of absence and is studying at Vanderbilt University. Mr. PHIL SHEFFEY, of Maryville College, is taking his place for the year.

At Mississippi College, Clinton, Miss., Mr. J. R. HITT has been promoted to a full professorship, and Mr. C. A. LOVELL has been appointed instructor of mathematics.

At St. Ignatius College, Cleveland, Ohio, Professor E. J. O'LEARY has been transferred to St. Louis University, and Mr. H. P. HECKEN has been appointed professor of mathematics.

Miss EMMA L. KONANTZ, associate professor of mathematics at Ohio Wesleyan University, has accepted a permanent appointment at Peking University, effective the second semester of this year. Professor Konantz, while on leave of absence 1919–1921, taught in Peking University, and collaborated with Professor CHEN in his work on "History of Chinese Mathematics." (See *1922*, 276.)

Dr. M. A. NORDGAARD, of Columbia University, has been appointed professor of mathematics and head of the department at Antioch College, Yellow Springs, Ohio.

Mr. WAYNE DANCER, formerly a teaching fellow, has been appointed instructor of mathematics at Toledo University.

At the University of Notre Dame, Mr. DANIEL HULL has been promoted to an assistant professorship, and Mr. W. L. SHILTS, of Notre Dame, has been appointed instructor of mathematics.

At Hanover College, Hanover, Ind., Professor C. A. REAGAN has resigned to accept a position as field agent of Missouri Valley College, Marshall, Mo., and Miss LENA R. COLE, of the University of Missouri, has been appointed to fill the vacancy.

Mr. A. O. BOATMAN, recently a graduate student at De Pauw University and the University of Indiana, has been appointed professor of physics at Carthage College, Carthage, Ill. His duties include assisting in the department of mathematics.

Mr. BURGOYNE GRIFFING, of the University of Kansas, has been added to the staff of the mathematics department of Des Moines University.

At Beloit College, Beloit, Wis., Professor W. A. HAMILTON has been appointed chairman of the Administrative Committee, which is in charge of the college during the interim while a new president is being sought to replace President M. A. BRANNON, recently appointed Chancellor of the University of Montana. As this work requires all of Professor HAMILTON's time, Professor H. H. CONWELL has been appointed acting head of the department of mathematics.

Mr. J. M. JACOBY has been appointed instructor of mathematics at St. Mary's College, St. Mary's, Kansas.

Mr. DON BROUSE, of Purdue University, has been appointed assistant professor of mathematics at Baker University, Baldwin, Kansas.

At Kansas State Agricultural College, Mr. W. H. ROWE, Mr. W. C. JANES, and Miss THIRZA A. MOSSMAN have been appointed instructors of mathematics.

Miss LAURA DUERNER, of the University of Oregon, has been appointed instructor of mathematics at North Dakota Agricultural College.

Mr. C. R. HILLARD, a graduate of Franklin and Marshall College, has been appointed head of the mathematics department in the College of the Ozarks, Clarksville, Ark.

Dean W. A. BARTLETT, of Pomona (Cal.) Junior College, has been elected president of the mathematics section of the Southern California Teachers' Association.

Professor C. G. DARWIN, who has been lecturing this year at the California Institute of Technology, has been appointed to the newly instituted Tait chair of Natural Philosophy at the University of Edinburgh.

At the University of Georgia, Professor D. F. BARROW presented a paper "On square roots of integers" to the Georgia Academy of Science at the annual meeting in December. Professor R. P. STEPHENS was elected president of the Academy.

The Minister of Education of France has approved a regulation admitting as candidates for the degree of Doctor of Science, students possessing certain foreign degrees, regarded as equivalent or superior to the "diplôme de licence." Amongst these are candidates from those American universities belonging to the Association of American Universities who possess a master's or doctor's degree, or a certificate stating that they have completed at least two years of graduate study leading to the Ph.D. Such candidates need not present the diplôme de licence.

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WITH THE COÖPERATION OF

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E. L. DODD	D. C. GILLESPIE	F. D. MURNAGHAN
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